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SSIT

Apuntes de clase

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SSIT

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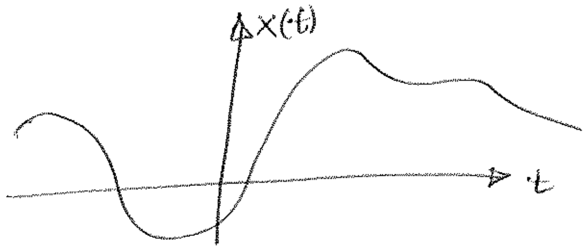
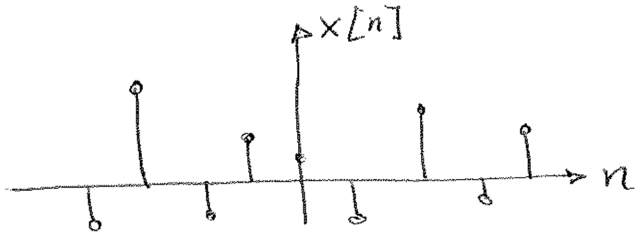
Tema 1: Análisis en el dominio del tiempo

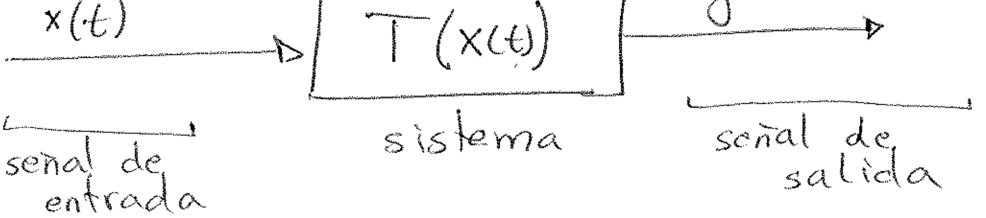
Tiempo continuo:
señal continua
$t \in \mathbb{R}$
$x(t) \in \mathbb{R}$



Tiempo discreto:
señal discreta
$n \in \mathbb{Z}$
$x[n] \in \mathbb{R}$



1. Concepto de señal
2. Concepto de sistema
3. Operaciones



Operaciones con señales:

suma: $z(t) = a\,x(t) + b\,y(t)$ // $z[n] = a\,x[n] + b\,y[n]$

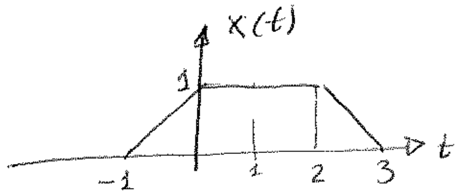
multiplicación: $z(t) = x(t)\cdot y(t)$ // $z[n] = x[n]\cdot y[n]$

Integración: $z(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ // $z[n] = \sum_{-\infty}^{n} x[k]$

derivación: $z(t) = \dfrac{d\,x(t)}{dt}$ // $z[n] = x[n] - x[n-1]$

Operaciones sobre la variable independiente:

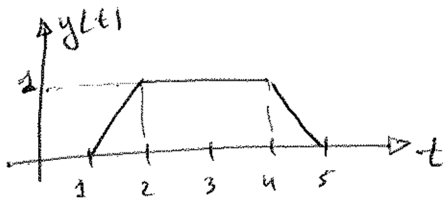
Ej:



$$x(t) = \begin{cases} t+1, & -1 \le t < 0 \\ 1, & 0 \le t < 2 \\ -t+3, & 2 \le t < 3 \\ 0, & \text{resto} \end{cases}$$

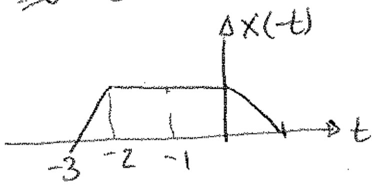
→ **Desplazamiento:** $y(t) = x(t - t_0)$

Ej: $y(t) = x(t-2) = \begin{cases} (t-2)+1, & -1 \le t-2 < 0 \equiv 1 \le t < 2 \\ 1, & 0 \le t-2 < 2 \equiv 2 \le t < 4 \\ -(t-2)+3, & 2 \le t-2 < 3 \equiv 4 \le t < 5 \\ 0, & \text{resto} \equiv \text{resto} \end{cases}$



→ **Reflexión:** $y(t) = x(-t)$

Ej: $y(t) = x(-t) = \begin{cases} (-t)+1, & -1 \le -t < 0 \equiv 0 < t < 1 \\ 1, & 0 \le -t < 2 \equiv -2 < t < 0 \\ -(-t)+3, & 2 \le -t < 3 \equiv -3 \le t < -2 \\ 0, & \text{resto} \equiv \text{resto} \end{cases}$



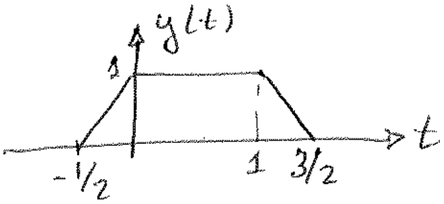
→ **Escalado:** $y(t) = x(at)$; $a \in \mathbb{R}^+$

si $a > 1 \Rightarrow$ compresión

si $0 < a < 1 \Rightarrow$ expansión

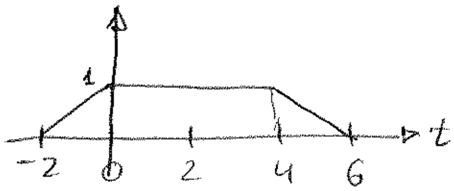
↳ compresión:

Ej: $y(t) = x(2t) = \begin{cases} 2t+1, & -1 \le 2t < 0 & \equiv -\frac{1}{2} \le t < 0 \\ 1, & 0 \le 2t < 2 & \equiv 0 \le t < 1 \\ -2t+3, & 2 \le 2t < 3 & \equiv 1 \le t < 3/2 \\ 0, & \text{resto} & \equiv \text{resto} \end{cases}$



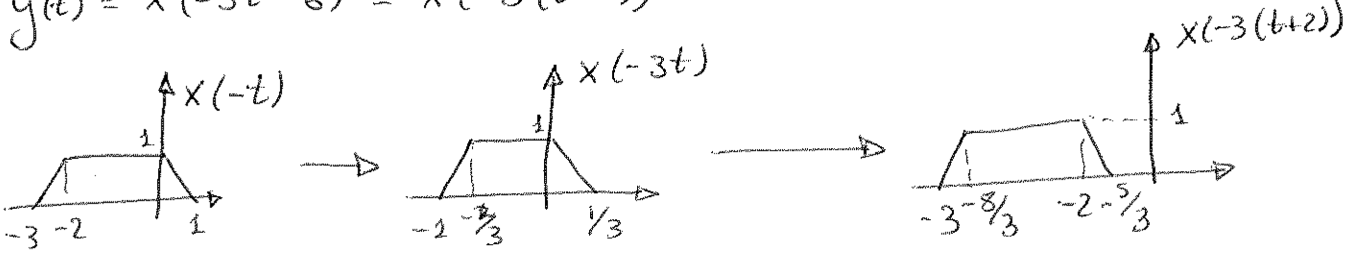
↳ expansión:

Ej: $y(t) = x(\frac{t}{2})$



→ General: $y(t) = x(\alpha t + \beta) = x(+\alpha(t + \frac{\beta}{\alpha}))$ (1°, 2°, 3°) — se opera en orden de fuera a dentro

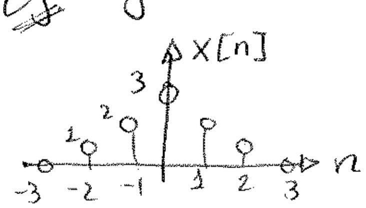
Ej: $y(t) = x(-3t-6) = x(-3(t+2))$

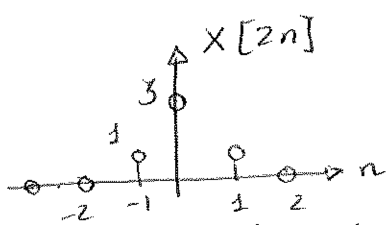


Ídem para discreto!

No obstante, ojo con el escalado en discreto:

Ej: $y[n] = x[2n]$





perdemos valores!

$y[-2] = x[-4] = 0$
$y[-1] = x[-2] = 1$
$y[0] = x[0] = 3$
$y[1] = x[2] = 1$
$y[2] = x[4] = 0$

Ej: $y[n] = x[\frac{n}{2}]$ no se pueden tomar valores en n impar, necesitamos evaluar sólo la señal en números n múltiplos de 2: $n = \dot{2}$

$$y[n] = \begin{cases} x[\frac{n}{2}], & n = \dot{2} \\ 0, & n \ne \dot{2} \end{cases}$$

Señales básicas en tiempo continuo:

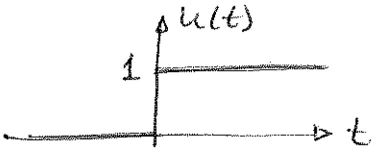
→ $x(t) = A\cos(w_0 t + \phi)$

→ $x(t) = c \cdot e^{at} \quad ; \quad c \in \mathbb{C} \,,\, a \in \mathbb{C}$

$$x(t) = |c|\, e^{j\theta} \cdot e^{(\sigma + jw_0)t}$$
$$= |c|\, e^{\sigma t + j(w_0 t + \theta)}$$
$$= |c|\, e^{\sigma t}\left[\cos(w_0 t + \theta) + j\,\mathrm{sen}(w_0 t + \theta)\right]$$

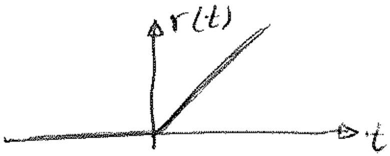
→ función escalón: $u(t)$

$$u(t) = \begin{cases} 0 \,, & t < 0 \\ 1 \,, & t > 0 \end{cases}$$



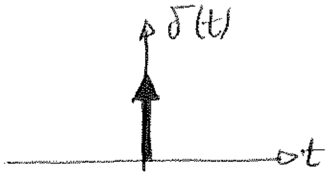
→ función rampa: $r(t)$

$$r(t) = \begin{cases} 0 \,, & t < 0 \\ t \,, & t \geq 0 \end{cases}$$

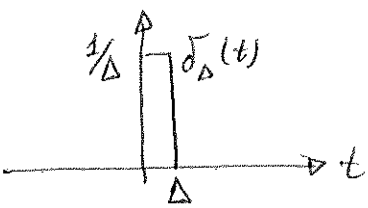


$$r(t) = t \cdot u(t)$$
$$u(t) = \frac{d\,r(t)}{dt}$$

→ Delta de Dirac



$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \end{cases}$$



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$x(t) \cdot \delta(t) = x(0)\,\delta(t)$$
$$x(t) \cdot \delta(t-a) = x(a)\,\delta(t-a)$$
$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{+\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{+\infty}\delta(t-t_0)\,dt = x(t_0)$$
$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0 \,, & t < 0 \\ 1 \,, & t > 0 \end{cases} = u(t) \Leftrightarrow \delta(t) = \frac{du(t)}{dt}$$

Señales básicas en tiempo discreto.

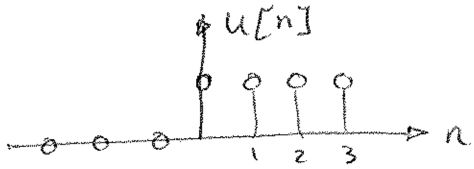
→ $x[n] = A\cos(w_0 n + \phi)$

→ $x[n] = c\, e^{a \cdot n}$; $c \in \mathbb{C}$, $a \in \mathbb{C}$

$$x[n] = |c|\, e^{j\theta}\, e^{(\sigma + jw_0)n}$$
$$= |c|\, e^{\sigma n}\, e^{j(w_0 n + \theta)}$$
$$= |c|\, e^{\sigma n}\left[\cos(w_0 n + \theta) + j\,\mathrm{sen}(w_0 n + \theta)\right]$$

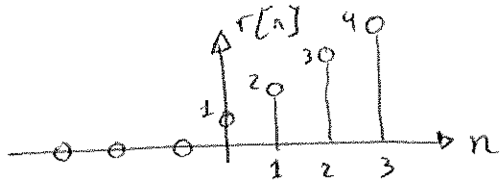
→ función escalón: $u[n]$

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



→ función rampa: $r[n]$

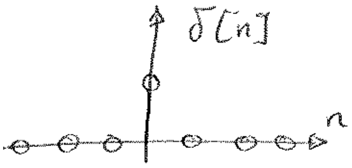
$$r[n] = \begin{cases} n+1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$r[n] = \sum_{k=-\infty}^{n} u[n]$$

$$u[n] = r[n] - r[n-1]$$

→ Delta de Kronecker



$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$x[n] \cdot \delta[n] = x[0]\,\delta[n]$$

$$x[n] \cdot \delta[n-n_0] = x[n_0] \cdot \delta[n-n_0]$$

$$\sum_{n=-\infty}^{+\infty} x[n] \cdot \delta[n-n_0] = x[n_0]$$

$$u[n] - u[n-1] = \delta[n]$$

$$\sum_{k=-\infty}^{n} \delta[k] = u[n]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Características y parámetros asociados

→ Valor de pico: $x_p = \max\{|x(t)|\}$

$x_p = \max\{|x[n]|\}$

→ Valor medio: $<x(t)> = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$

$<x[n]> = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} x[n]$

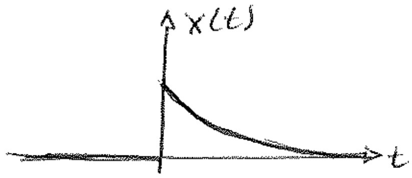
→ Energía: $E_x = \lim_{T\to\infty}\int_{-T}^{+T} |x(t)|^2\,dt$

$E_x = \lim_{N\to\infty}\sum_{n=-N}^{+N} |x[n]|^2$

→ Potencia media: $P_m = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$

$P_m = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2$

Ej: $x(t) = e^{-at}u(t)\ ;\ a>0$



$<x(t)> = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,dt =$

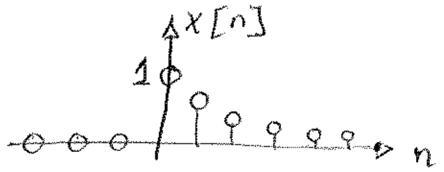
$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-at}u(t)\,dt =$

$= \lim_{T\to\infty}\frac{1}{2T}\int_{0}^{T} e^{-at}\,dt =$

$\lim_{T\to\infty}\frac{e^{-at}}{-2Ta}\Big|_0^T = \lim_{T\to\infty}\frac{1}{2Ta}(1-e^{-aT}) = 0$

$E_x = \lim_{T\to\infty}\int_{-T}^{T} x^2(t)\,dt = \lim_{T\to\infty}\int_{0}^{T} e^{-2at}\,dt = \lim_{T\to\infty}\frac{1}{2a}(1-e^{-2aT}) = \frac{1}{2a}$

Ej: $x[n] = a^n u[n] \ , \ a \in (0,1)$



$$\langle x[n] \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} x[n] =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{0}^{N} a^n = \left\{ \begin{array}{c} \text{serie} \\ \text{geom.} \end{array} \right\} =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot \frac{a^0 - a^N \cdot a}{1-a} =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot \frac{1 - a^{N+1}}{1-a} = 0$$

Suma de serie aritmética

$$S = \frac{a_1 + a_n}{2} \cdot n$$

Suma de serie geométrica

$$S = \frac{a_0 - a_n \cdot r}{1-r}$$

$$E_x = \lim_{N\to\infty} \sum_{-N}^{N} a^{2n} u[n] =$$

$$= \lim_{N\to\infty} \sum_{0}^{N} a^{2n} = \left\{ \begin{array}{c} \text{serie} \\ \text{geom} \end{array} \right\} =$$

$$= \lim_{N\to\infty} \frac{a^0 - a^{2(N+1)}}{1-a^2} = \frac{1}{1-a^2}$$

Periodicidad:

$x(t) = x(t+T) \Leftrightarrow$ periódica

$x[n] = x[n+N] \Leftrightarrow$ periódica

$x_1(t) \Rightarrow T_1 \rightarrow x_1(t) = x_1(t + k_1 T_1)$

$x_2(t) \Rightarrow T_2 \rightarrow x_2(t) = x_2(t + k_2 T_2)$

$x_3(t) = x_1(t) + x_2(t)$

$x_3(t+T) = x_1(t+T) + x_2(t+T) = x_3(t) \Leftrightarrow T = k_1 T_1 = k_2 T_2$

$$\frac{k_1}{k_2} = \frac{T_2}{T_1}$$

$$T = mcm(T_1, T_2)$$

$x(t) = e^{jw_0 t} = \cos(w_0 t) + j\,\text{sen}(w_0 t)$

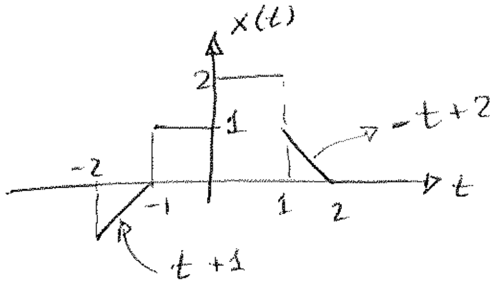
$x(t+T) = e^{jw_0 (t+T)} = e^{jw_0 t} e^{jw_0 T} = x(t) \Leftrightarrow w_0 T = 2k\pi$

$x[n] = e^{jw_0 n}$

$x[n+N] = e^{jw_0 (n+N)} = e^{jw_0 n} e^{jw_0 N} = x[n] \Leftrightarrow w_0 N = 2k\pi$

$\left\{\right.$ no existe todo valor de w_0 y k que hagan $N \in \mathbb{Z}$

Ej: 1.21

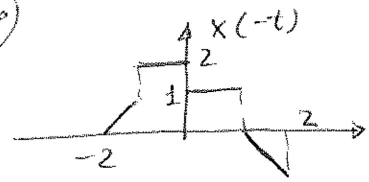


d) $y(t) = x(4 - t/2)$

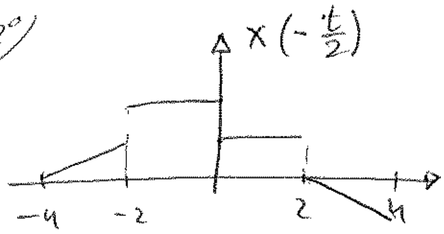
$= x\left(-\frac{1}{2}(t-8)\right)$

1° 2° 3°

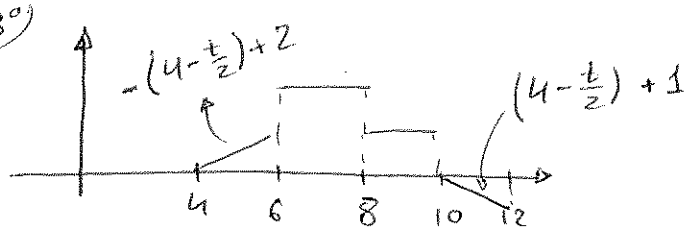
1°)



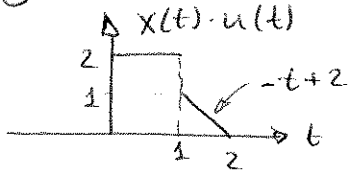
2°)

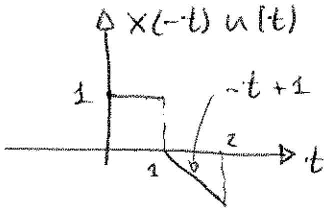


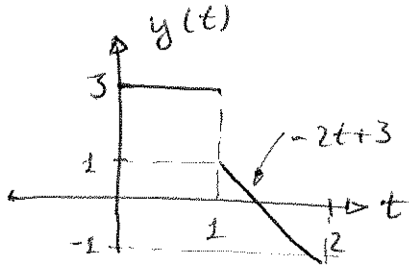
3°)



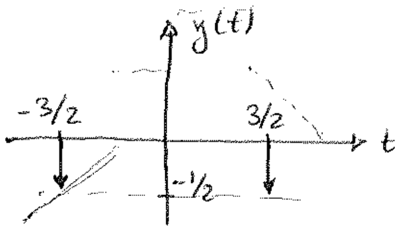
e) $y(t) = (x(t) + x(-t))\, u(t)$



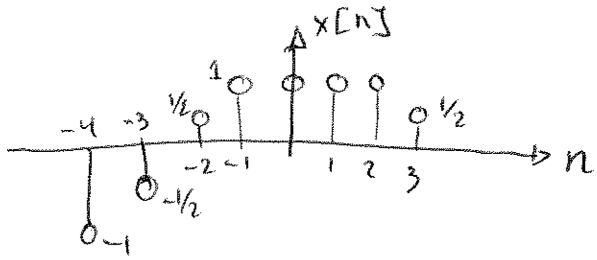




f) $y(t) = x(t)\left[\delta\left(t+\frac{3}{2}\right) - \delta\left(t-\frac{3}{2}\right)\right]$



Ej: 1.22

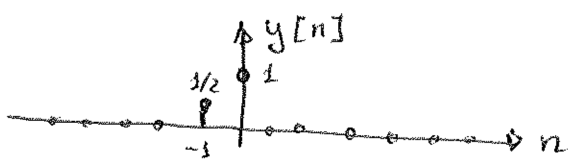


d) $y[n] = x[3n+1]$

$y[0] = x[1] = 1$

$y[1] = x[4] = 0$

$y[-1] = x[-2] = 1/2$



e) $y[n] = x[n]\underbrace{u[3-n]}$

$$\begin{cases} 1, & 3-n \geq 0 \Leftrightarrow n \leq 3 \\ 0, & \text{resto} \end{cases} \qquad y[n] = x[n]$$

f) $y[n] = x[n-2]\,\delta[n-2]$

$$y[n] = x[n-2]\Big|_{n=2} \cdot \delta[n-2] = x[0]\cdot\delta[n-2] = \delta[n-2]$$

Simetría de señales:

señal par $\Leftrightarrow x(t) = x(-t) \Leftrightarrow$ señal simétrica

señal impar $\Leftrightarrow x(t) = -x(-t) \Leftrightarrow$ señal antisimétrica

señal par $\Leftrightarrow x(t) = x^*(-t) \Leftrightarrow$ señal hermítica

señal impar $\Leftrightarrow x(t) = -x^*(-t) \Leftrightarrow$ señal antihermítica

siendo x^* el complejo conjugado de x

Ej 1.25:

c) $x(t) = \left[\cos\left(2t - \frac{\pi}{3}\right)\right]^2 = \dfrac{1 + \cos\left(2\left(2t - \frac{\pi}{3}\right)\right)}{2} = \dfrac{1}{2} + \dfrac{1}{2}\cos\left(4t - \frac{2\pi}{3}\right)$

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t}\cdot e^{j\omega_0 T} = e^{j\omega_0 t} \Leftrightarrow e^{j\omega_0 T} = 1 \Leftrightarrow T = \frac{2k\pi}{\omega_0}$$

$\omega_0 = 4 \Rightarrow T = k\frac{\pi}{2}$ si $k = 1$ (periodo fundam.) $\Rightarrow T_0 = \frac{\pi}{2}$

Ej 1.26:

a) $x[n] = \text{sen}\left(\frac{6\pi}{7}n + 1\right)$

$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \Leftrightarrow N = k\frac{2\pi}{\omega_0}$; $N \in \mathbb{N} \Rightarrow \omega_0 = q\cdot\pi$, $q \in \mathbb{Q}$

$N = k\frac{2\pi}{6\pi}7 = k\frac{1}{3}7 \Rightarrow$ (periodo fundam.) $k = 3 \Rightarrow N_0 = 7$

b) $x[n] = \cos\left(\frac{n}{8} - \pi\right)$

$N = k \cdot 2\pi \cdot 8 \Rightarrow$ No periódico, ya que no podemos encontrar un k que haga N válido

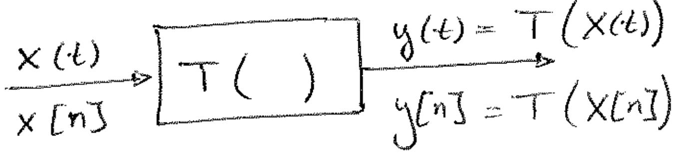
d) $x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right) = \frac{1}{2}\left[\underbrace{\cos\left(\frac{3\pi}{4}n\right)}_{\text{periódica}} + \underbrace{\cos\left(\frac{\pi}{4}n\right)}_{\text{periódica}}\right]$

$N = m.c.m.(N_1, N_2)$; $N_1 = k\frac{2\pi}{3\pi}\cdot 4 \rightarrow k=3 \Rightarrow N_1 = 8$

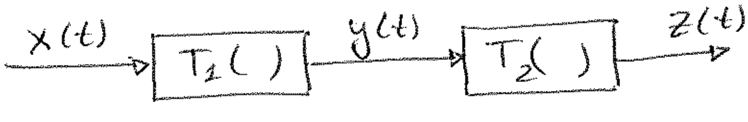
$N_2 = k\frac{2\pi}{\pi}\cdot 4 \rightarrow k=1 \Rightarrow N_2 = 8$

$N = m.c.m(8, 8) = 8$

Concepto de sistema. Interconexión

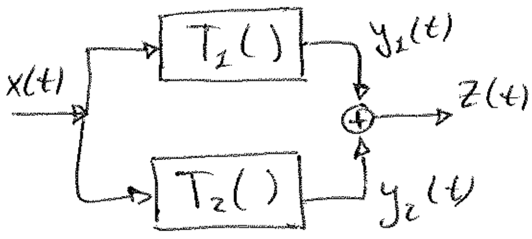


→ conexión serie o en cascada:



$z(t) = T_2(y(t)) = T_2(T_1(x(t)))$

→ conexión paralelo:



$z(t) = y_1(t) + y_2(t) = T_1(x(t)) + T_2(x(t))$

Propiedades de los sistemas.

* **Memoria:** un sistema no tiene memoria cuando la salida en un instante sólo depende de la entrada en ese instante.

  $x(t) \longrightarrow y(t) = x^2(t)$ no tiene memoria

  $x[n] \longrightarrow y[n] = \sum_{-\infty}^{N} x[n]$ sí tiene memoria

* **Causalidad:** un sistema es causal si la salida en un instante depende sólo de su pasado y presente relativo. No del futuro

  $y[n] = \underbrace{x[n]}_{\text{presente}} + \underbrace{x[n-3]}_{\text{pasado}}$ sí causal

  $y[n] = x[n+3]$ (futuro) No causal

  si depende sólo del futuro $\equiv$ anticausal

* **Estabilidad:** un sistema es estable si para toda señal de entrada acotada produce una salida también acotada

  $\forall x(t),\ |x(t)| < \alpha < \infty,\ \forall t \Rightarrow |y(t)| < \beta < \infty\ \forall t$

  $y[n] = \sum_{-\infty}^{n} u[k] = r[n]$ No estable

* **Invarianza temporal:** un sistema es invariante en el tiempo si desplazamientos en la señal de entrada produce desplazamientos en la salida del mismo orden

  $$\left.\begin{array}{l} x_0(t) \longrightarrow y_0(t) \\ x_1(t) = x_0(t-t_0) \longrightarrow y_1(t) = y_0(t-t_0) \end{array}\right\} \text{invar. temp.}$$

  Ej: $y_1(t) = t x_1(t) = t\, x_0(t-t_0) \neq y_0(t-t_0) = (t-t_0)\, x_0(t-t_0)$

  variante

Linealidad:

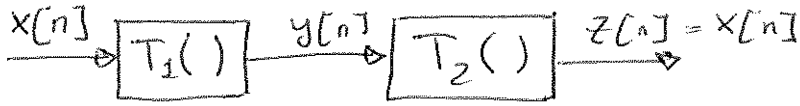
$x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

$$\left.\begin{aligned} \text{si } x_3(t) &= \alpha x_1(t) + \beta x_2(t) \\ y_3(t) &= \alpha y_1(t) + \beta y_2(t) \end{aligned}\right\} \text{ lineal}$$

Invertibilidad:

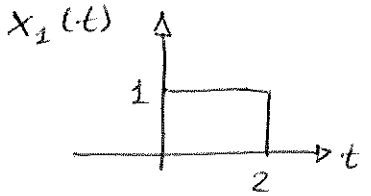
un sistema es invertible si con la señal de salida podemos definir de forma unequívoca la señal de entrada.

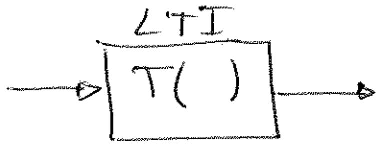


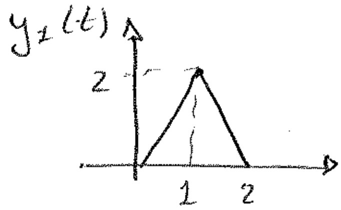
si $Z[n] = x[n] \Rightarrow T_2$ es el sistema inverso de T_1

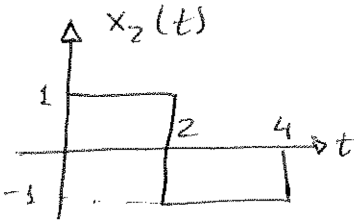
LTI: Lineal and Time Invariant

Ej 1.31:





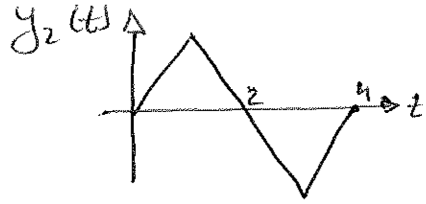


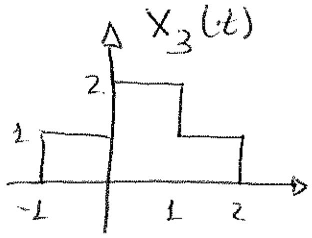


$x_2(t) = x_1(t) - x_1(t-2)$

$y_2(t) = \{\text{lineal}\} = T(x_1(t)) - T(x_1(t-2))$

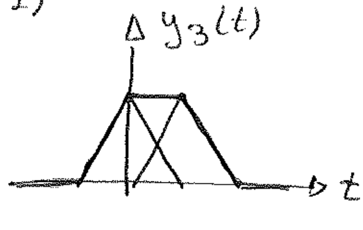
$y_2(t) = y_1(t) - y_1(t-2)$





$x_3(t) = x_1(t) + x_1(t+1)$

$y_3(t) = y_1(t) + y_1(t+1)$



Ej 1.43 a

$$x(t) \longrightarrow \boxed{T(\ )} \longrightarrow y(t) \qquad \text{LTI}$$

$x(t)$ periódica de periodo $T \rightsquigarrow y(t)$??

$$x(t) = \sum_{k=-\infty}^{\infty} x_p(t-kT) \; ; \quad x_p(t) = 1 \text{ periodo de } x(t)$$

$$y(t) = T(x(t)) = T\left(\sum_{k=-\infty}^{+\infty} x_p(t-kT)\right) = \sum_{k=-\infty}^{+\infty} T(x_p(t-kT)) =$$

$$= \sum_{k=-\infty}^{+\infty} y_p(t-kT) \; ; \quad y_p(t) = T(x_p(t))$$

Sistemas definidos por EDOs:

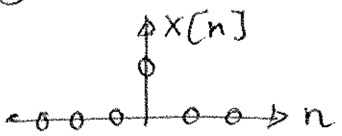
$$x(t) \longrightarrow \boxed{\sum_{k=0}^{M} a_k \frac{d y^k(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d x^k(t)}{dt^k}} \longrightarrow y(t)$$

$$x[n] \longrightarrow \boxed{\sum_{k=0}^{M} a_k \, y[n-k] = \sum_{k=0}^{N} b_k \, x[n-k]} \longrightarrow y[n]$$

En este curso no podremos resolverlas a no ser que forcemos las condiciones de reposo iniciales (CRI)

CRI $\Rightarrow$ si no hay entrada no hay salida $\Rightarrow$ LTI

Ej: $y[n] - \frac{1}{2} y[n-1] = x[n] = \delta[n]$

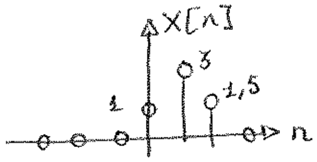


CRI $\rightarrow$ $y[n] = 0 \quad \forall n < 0$

$$\left.\begin{aligned}
y[0] &= \tfrac{1}{2} y[-1] + x[0] = 1 \\
y[1] &= \tfrac{1}{2} y[0] + x[1] = \tfrac{1}{2} \\
y[2] &= \tfrac{1}{2} y[1] + x[2] = \tfrac{1}{4} \\
y[3] &= \tfrac{1}{2} y[2] + x[3] = \tfrac{1}{8} \\
&\vdots
\end{aligned}\right\} \quad y[n] = \left(\tfrac{1}{2}\right)^n \cdot u[n]$$

Sistemas discretos: Respuesta al impulso

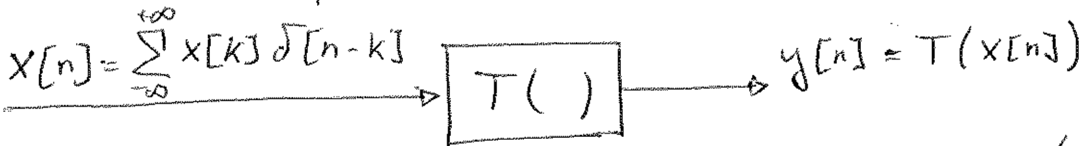
↳ Representación de una señal mediante una combinación de deltas (impulsos)



$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

↳ Calcular la respuesta de un sistema LTI



$$y[n] = T(x[n]) = T\Big(\sum_k x[k]\,\delta[n-k]\Big) = \sum_k x[k]\cdot T(\delta[n-k])$$

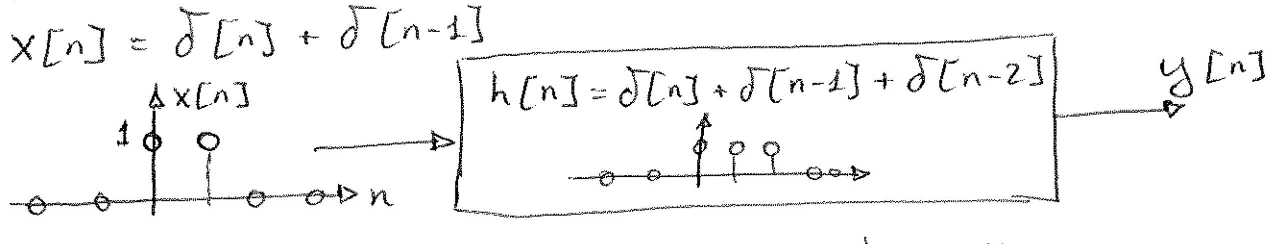
$$T(x[n]=\delta[n]) = h[n] = \text{respuesta al impulso}$$

$$y[n] = \sum_k x[k]\cdot h[n-k] = x[k] * h[n]$$

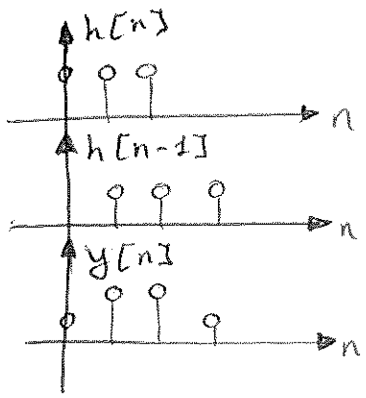
↑ operador convolución

Operador convolución [discreto]

$$x[n] = \delta[n] + \delta[n-1]$$

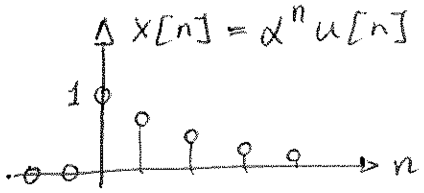


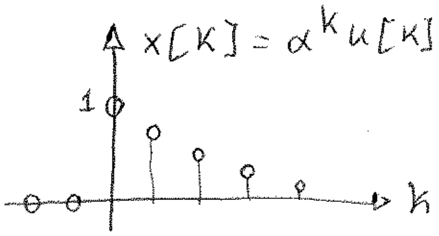
$$y[n] = \underbrace{\sum_k x[k]\cdot h[n-k]}_{x[k] * h[n]} = x[0]h[n] + x[1]\,h[n-1]$$



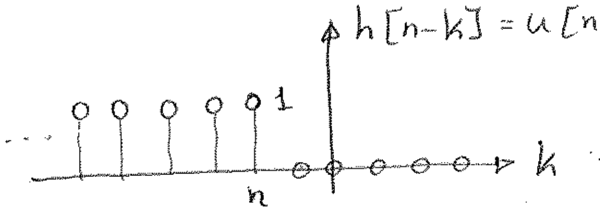
Ej:

$$\left.\begin{array}{l} x[n] = \alpha^n u[n],\ \alpha \in (0,1) \\ h[n] = u[n] \end{array}\right\} \quad \begin{array}{l} y[n] = x[n] * h[n] \\ y[n] = \sum_{k=-\infty}^{+\infty} x[k]\cdot h[n-k] \end{array}$$





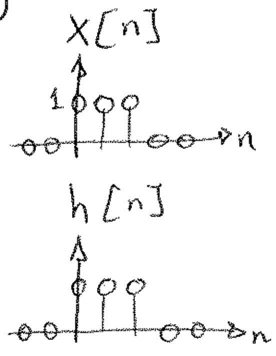
$h[n-k] = u[n-k] = 1 \Leftrightarrow n-k \geq 0$
$n \geq k$



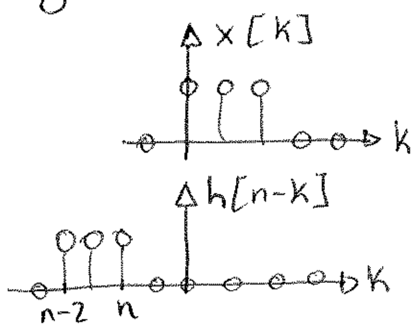
$$\begin{cases} \text{si } n<0 \Rightarrow y[n] = \sum_{k} x[k]\cdot h[n-k] = 0 \\ \text{si } n=0 \Rightarrow y[n] = x[0]\cdot h[n-0] = 1 \\ \text{si } n=1 \Rightarrow y[n] = x[0]\cdot h[n] + x[1]\, h[n-1] = 1+\alpha \\ \quad\vdots \\ n \geq 0 \Rightarrow y[n] = \sum_{k=0}^{n} x[k]\, h[n-k] = \\ \qquad = \sum_{k=0}^{n} \alpha^k \cdot 1 = \dfrac{1-\alpha^{n+1}}{1-\alpha} \end{cases}$$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha}\cdot u[n]$$

Ej:



$$y[n] = x[n] * h[n] = \sum_{k} x[k]\, h[n-k]$$



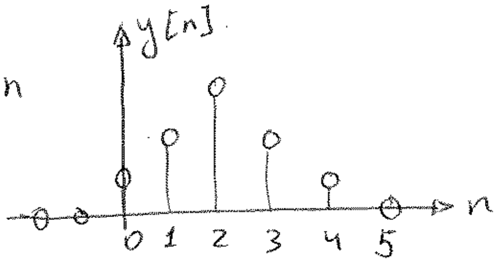
$n<0 \Rightarrow y[n] = 0$
$n=0 \Rightarrow y[n] = 1$
$n=1 \Rightarrow y[n] = 2$
$n=2 \Rightarrow y[n] = 3$
$n=3 \Rightarrow y[n] = 2$
$n=4 \Rightarrow y[n] = 1$
$n \geq 5 \Rightarrow y[n] = 0$

$n<0 \Rightarrow y[n] = 0$

$0 \leq n \leq 2 \Rightarrow y[n] = \sum_{0}^{n} 1\cdot 1 = n+1$

$1 \leq n-2 \leq 2 \Rightarrow y[n] = \sum_{n-2}^{2} 1\cdot 1 = 2-(n-2)+1 = 5-n$

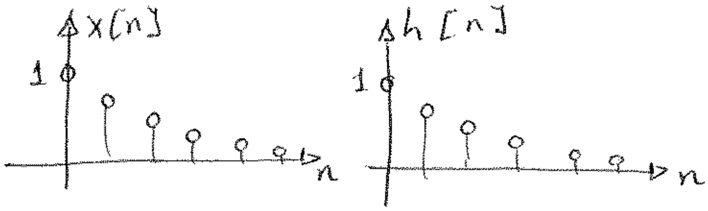
$n-2 > 2 \Rightarrow y[n] = 0$



Ej 2.21

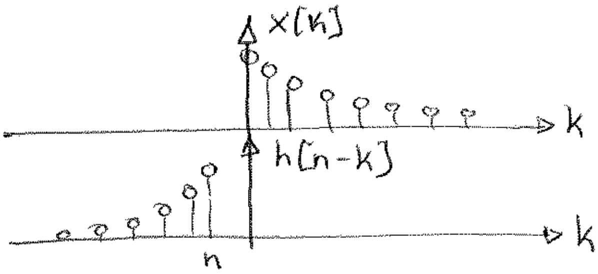
a) $x[n] = \alpha^n u[n]$, $\alpha, \beta \in (0,1)$

$h[n] = \beta^n u[n]$, $\alpha \neq \beta$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$h[n-k] = \beta^{n-k} \underbrace{u[n-k]}_{=1 \Leftrightarrow n-k \geq 0 \Leftrightarrow n \geq k}$$



$n < 0 \Rightarrow y[n] = 0$

$n \geq 0 \Rightarrow y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k}$

$$n \geq 0 \Rightarrow y[n] = \sum_{k=0}^{n} \alpha^k \beta^n \beta^{-k} = \beta^n \sum_{k=0}^{n} \alpha^k \beta^{-k} = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k$$

$$= \beta^n \left[\frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1}}{1 - \alpha/\beta} \right]$$

$$y[n] = \beta^n \cdot \frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1}}{1 - \alpha/\beta}\, u[n] \leftarrow \text{si } \alpha \neq \beta$$

si $\alpha = \beta$

$$n \geq 0 \Rightarrow y[n] = \sum_{k=0}^{n} \alpha^k \beta^n \beta^{-k} = \beta^n \sum_{k=0}^{n} \alpha^k \beta^{-k} = \beta^n \sum_{k=0}^{n} 1^k = \beta^n (n+1)$$

$(\alpha = \beta)$

$$y[n] = \beta^n (n+1)\, u[n] \leftarrow \text{si } \alpha = \beta$$

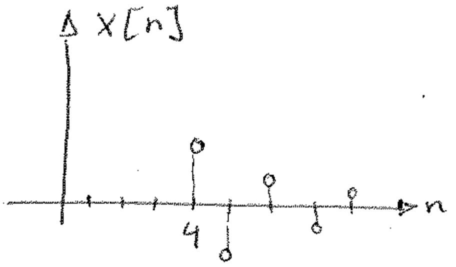
Ej 2.21

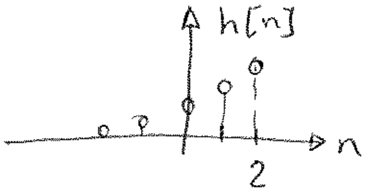
a) $x[n] = \left(\frac{-1}{2}\right)^n u[n-4]$

$h[n] = 4^n \, u[\underbrace{2-n}]$

$2-n \geq 0$

$2 \geq n$



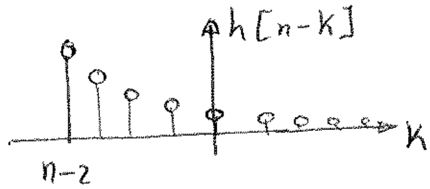


$$y[n] = \sum x[k] \; h[n-k]$$

$\hookrightarrow 4^{n-k} \, u[\underbrace{2-(n-k)}]$

$2-n+k \geq 0$

$k \geq n-2$

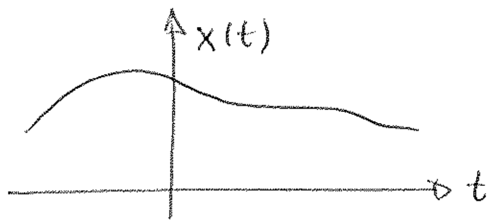


si $n-2 \leq 4$, $\displaystyle y[n] = \sum_{4}^{\infty} \left(\frac{-1}{2}\right)^k 4^{n-k}$

$$y[n] = 4^n \sum_{4}^{\infty} \left(\frac{-1}{8}\right)^k = 4^n \frac{\left(\frac{-1}{8}\right)^4}{1+1/8}$$

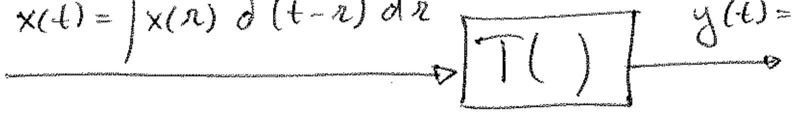
si $n-2 > 4$, $\displaystyle y[n] = 4^n \sum_{n-2}^{\infty} \left(\frac{-1}{8}\right)^k = 4^n \frac{\left(\frac{-1}{8}\right)^{n-2}}{1+1/8}$

Sistemas continuos: Respuesta al impulso



$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$x(t) = \int x(\tau)\, \delta(t-\tau)\, d\tau$

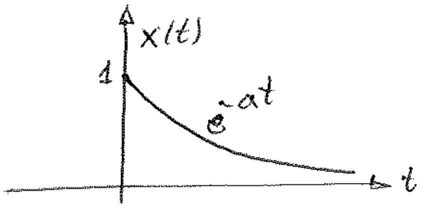


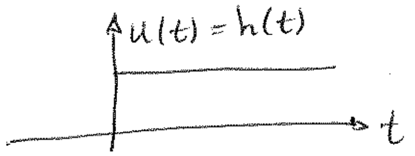
$$\begin{aligned}
y(t) &= T(x(t)) = T\left(\int x(\tau)\,\delta(t-\tau)\, d\tau\right) \\
&= \int T(x(\tau)\,\delta(t-\tau))\, d\tau \\
&= \int x(\tau)\, T(\delta(t-\tau))\, d\tau \\
&= \int x(\tau)\, h(t-\tau)\, d\tau \\
&= x(t) * h(t)
\end{aligned}$$

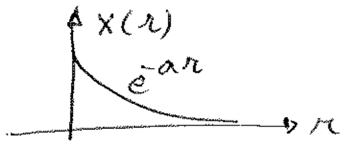
operador convolución

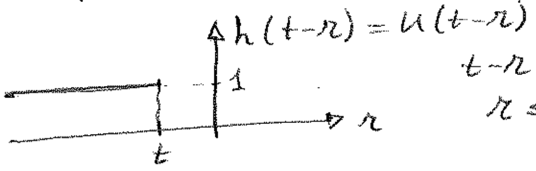
Ej: $x(t) = e^{-at}u(t), \quad a>0$
$h(t) = u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(r)\,h(t-r)\,dr$$









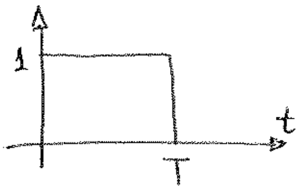
$t-r \geq 0$
$r \leq t \Rightarrow 1$

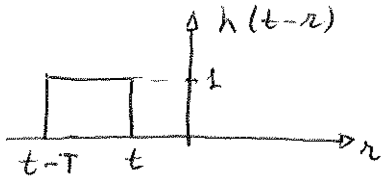
$t<0, \quad y(t)=0$

$t>0, \quad y(t) = \int_0^t x(r)\,dr = \int_0^t e^{-ar}\,dr = \frac{1}{-a}\left(e^{-at}-1\right)$

$$y(t) = \frac{1-e^{-at}}{a}\cdot u(t)$$

Ej: $x(t) = h(t)$ $\qquad y(t) = \int x(r)\,h(t-r)\,dr$



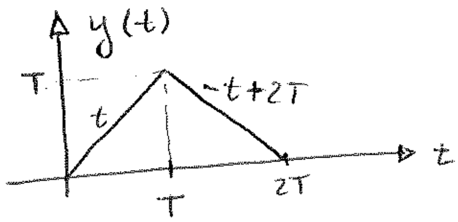


$t \leq 0, \quad y(t) = 0$

$0 < t \leq T, \quad y(t) = \int_0^t dr = t$

$0 < t-T \leq T, \quad y(t) = \int_{t-T}^{T} dr = T-(t-T) = -t+2T$
$T < t \leq 2T$

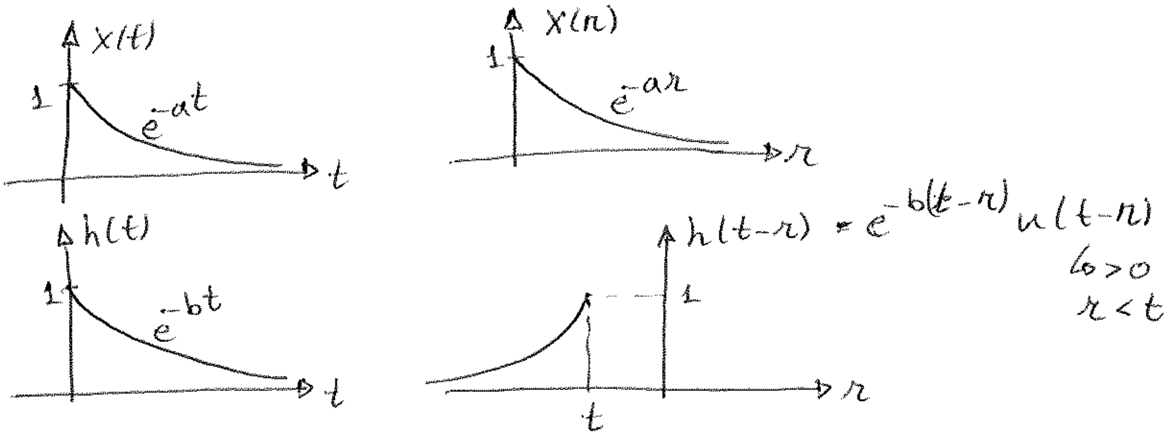
$t-T > T, \quad y(t) = 0$
$t > 2T$



Ej. 2.22

a) $x(t) = e^{-at} u(t)$

$h(t) = e^{-bt} u(t) \qquad a, b > 0, \ a \neq b$



$t<0, \ y(t) = 0$

$$t \geq 0, \ y(t) = \int_0^t e^{-a\tau} e^{-b(t-\tau)} \, d\tau = e^{-bt} \int_0^t e^{-(a-b)\tau} \, d\tau =$$

$$= e^{-bt} \frac{1}{a-b} e^{-(a-b)\tau} \Big|_0^t = \frac{e^{-bt}}{a-b} \left(e^{-(a-b)t} - 1 \right)$$

$$= \frac{\left(e^{-bt} - e^{-at}\right)}{a-b}$$

$$y(t) = \frac{e^{-bt} - e^{-at}}{a-b} \cdot u(t), \quad a \neq b$$

$$y(t) = t \cdot e^{-bt} \cdot u(t), \quad a = b$$

Propiedades del operador convolución:

→ Elemento neutro:

$x(t) * \delta(t) = x(t)$

$x[n] * \delta[n] = x[n]$

→ Desplazamiento:

$x(t) * \delta(t-t_0) = x(t-t_0) \neq x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

$x[n] * \delta[n-n_0] = x[n-n_0]$

→ Conmutativa

$$x(t) * y(t) = y(t) * x(t)$$

$$\int x(\tau)\, y(t-\tau)\, d\tau = \int y(\tau)\, x(t-\tau)\, d\tau$$

→ Asociativa

$$x(t) * \big(y(t) * z(t)\big) = \big(x(t) * y(t)\big) * z(t)$$

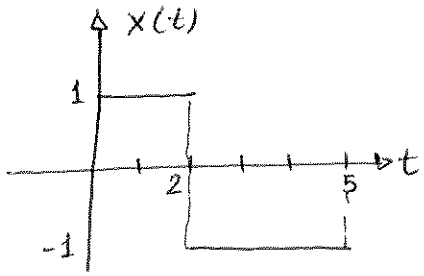
→ Distributiva respecto la suma

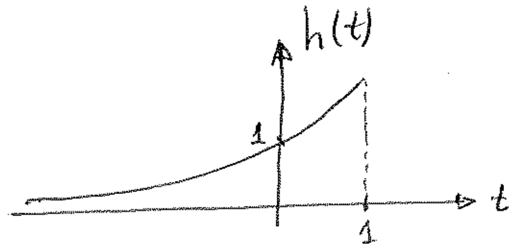
$$x(t) * \big(y_1(t) + y_2(t)\big) = x(t) * y_1(t) + x(t) * y_2(t)$$

Ej 2.22:

b) $x(t) = u(t) - 2u(t-2) + u(t-5)$

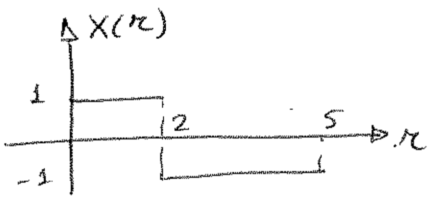
$h(t) = e^{2t} u(1-t)$

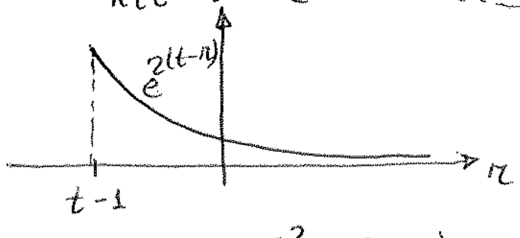




$$h(t-\tau) = e^{2(t-\tau)} \underbrace{u(1-t+\tau)}_{>0}$$

$\tau > t-1$





$t-1<0$, $y(t) = \int_0^5 x(\tau)\, h(t-\tau)\, d\tau = \int_0^2 e^{2(t-\tau)} d\tau - \int_2^5 e^{2(t-\tau)} d\tau$

$0 < t-1 \leq 2$, $y(t) = \int_{t-1}^2 e^{2(t-\tau)} d\tau - \int_2^5 e^{2(t-\tau)} d\tau$

$2 < t-1 \leq 5$, $y(t) = \int_{t-1}^5 e^{2(t-\tau)} d\tau$

$t-1 > 5$, $y(t) = 0$

Another way:

$$x(t) = u(t) * \delta(t) - 2\,u(t) * \delta(t-2) + u(t) * \delta(t-5)$$
$$= u(t) * \big(\delta(t) - 2\,\delta(t-2) + \delta(t-5)\big)$$

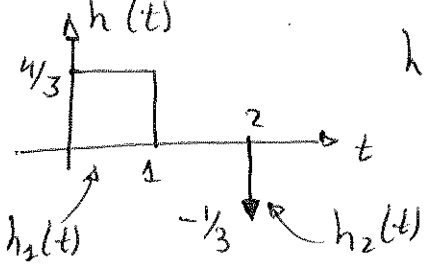
$$y(t) = x(t) * h(t) = u(t) * \big(\delta(t) - 2\delta(t-2) + \delta(t-5)\big) * h(t)$$

si $y_1(t) = u(t) * h(t)$

$$y(t) = y_1(t) - 2\,y_1(t-2) + y_1(t-5)$$

Ej 2.22

d) $x(t) = at + b$

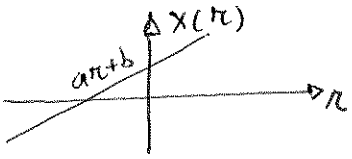


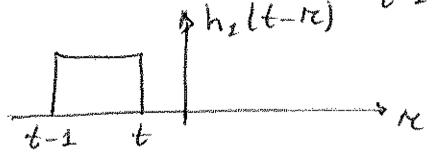
$$h(t) = h_1(t) + h_2(t)$$

$$y(t) = x(t) * \big(h_1(t) + h_2(t)\big) = y_1(t) + y_2(t)$$

$$y_2(t) = x(t) * \left(\frac{-1}{3}\right)\delta(t-2) = \frac{-1}{3}\,x(t-2) = \frac{-1}{3}\big(a(t-2) + b\big)$$

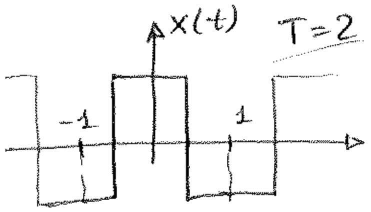
$$y_1(t) = x(t) * h_1(t) = \int x(\tau)\,h_1(t-\tau)\,d\tau = \frac{4}{3}\int_{t-1}^{t} a\tau + b\;d\tau$$

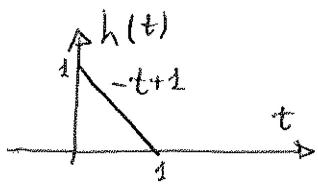


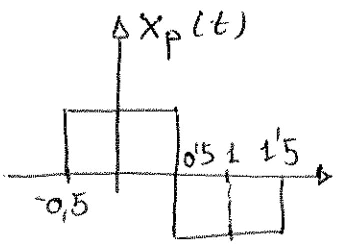


Ej. 2.22

e)







$$y(t) = x(t) * h(t) = \left(\sum_{k=-\infty}^{+\infty} x_p(t-kT)\right) * h(t)$$
$$= \left(\sum x_p(t) * \delta(t-kT)\right) * h(t) =$$
$$= x_p(t) * \sum \delta(t-kT) * h(t) =$$
$$= y_p(t) * \sum \delta(t-kT) = \sum y_p(t-kT)$$

Propiedades de los sistemas LTI:

→ Memoria: un sistema no tiene memoria si y sólo si:
$$h[n] = k\delta[n] \quad / \quad h(t) = k\delta(t)$$
es decir, no depende de sus versiones desplazadas

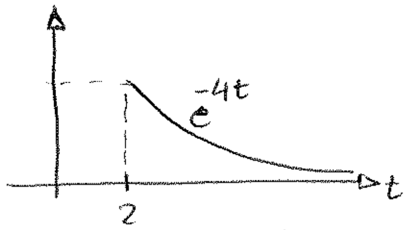
→ Causalidad: un sistema es causal si y sólo si:
$$h[n] = 0, \forall n<0 \quad / \quad h(t) = 0, \quad \forall t<0$$

→ Estabilidad: un sistema es estable si y sólo si:
$$\int |h(t)|\, dt < \infty \quad / \quad \sum |h[n]| < \infty$$

→ Invertibilidad: un sistema es invertible si y sólo si:
$$h[n] * h_i[n] = \delta[n] \quad / \quad h(t) * h_i(t) = \delta(t)$$

Ej. 2.29

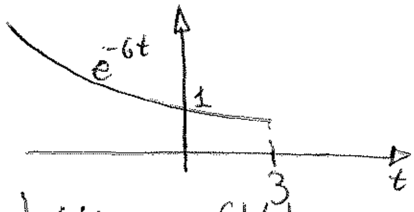
a) $h(t) = e^{-4t} u(t-2)$



$h(t) = 0, \forall t<0 \Rightarrow$ causal

$\int_2^{\infty} e^{-4t} dt < \infty \Rightarrow$ estable

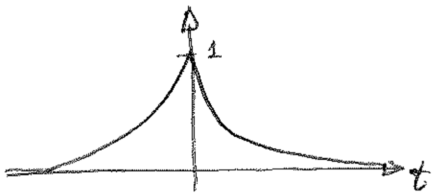
b) $h(t) = e^{-6t} u(3-t)$



$h(t) \neq 0, \forall t<0 \Rightarrow$ No causal

$\int_{-\infty}^{3} e^{-6t} dt = \infty \Rightarrow$ No estable

c) $h(t) = e^{-6|t|}$



$h(t) \neq 0 \ \forall t<0 \Rightarrow$ No causal

$\int_{-\infty}^{+\infty} e^{-6|t|} dt < \infty \Rightarrow$ estable

Tema 2: Transformada de Fourier de señales continuas

$$x(t)=e^{j\omega_0 t} \longrightarrow \boxed{\begin{matrix} h(t) \\ H(j\omega) \end{matrix}} \longrightarrow y(t)=x(t)*h(t)=\int x(t-\tau)\,h(\tau)\,d\tau$$

$$=\int e^{j\omega_0(t-\tau)}\,h(\tau)\,d\tau =$$

$$= e^{j\omega_0 t}\int e^{-j\omega_0 \tau}\,h(\tau)\,d\tau = e^{j\omega_0 t}\,H(j\omega_0)$$

Respuesta en frecuencia:

$$H(j\omega)=\int_{-\infty}^{+\infty} h(\tau)\,e^{-j\omega\tau}\,d\tau$$

Si la señal es No periódica usaremos la Transformada de Fourier
Si la señal es periódica usaremos el Desarrollo en Serie de Fourier

Transformada de Fourier:

$$X(j\omega)=\int_{-\infty}^{+\infty} x(t)\,e^{-j\omega t}\,dt \qquad x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,e^{j\omega t}\,d\omega$$

Ecuación de análisis ó Transformada directa

Ecuación de síntesis ó Transformada inversa

➡ Condiciones suficientes de existencia de la TF

- ↳ Absolutamente integrable $\int_{-\infty}^{+\infty} |x(t)|\,dt < \infty$
- ↳ Número finito de discontinuidades finitas en un intervalo finito
- ↳ Número finito de máximos y mínimos en un intervalo finito

(Condiciones de Dirichlet)

Ej: $X(j\omega_0) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega_0 t}\, dt$

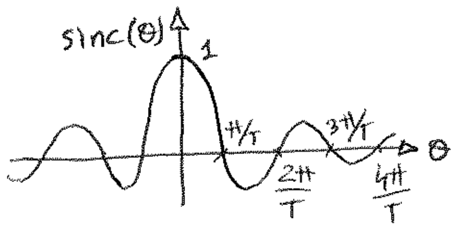
$$x(t) = \delta(t) \longrightarrow X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} \delta(t) \cdot \underbrace{e^{-j\omega \cdot 0}}_{1}\, dt = 1$$

$$x(t) = e^{-at}u(t),\ a>0\ ;\ X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(j\omega+a)t}\, dt$$
$$= \frac{1}{j\omega+a}\left(e^{-(j\omega+a)t}\right)\Big|_{\infty}^{0} = \frac{1}{j\omega+a}$$

$$x(t) = e^{-a|t|} \longrightarrow X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|}\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$x(t) = \begin{cases} 1, & -T<t<T \\ 0, & \text{resto} \end{cases} \longrightarrow X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{1}{j\omega}\left(e^{+j\omega T} - e^{-j\omega T}\right)$$
$$= \frac{2}{\omega}\, \frac{e^{j\omega T} - e^{-j\omega T}}{2j} = \frac{2}{\omega} \cdot \mathrm{sen}(\omega T) = 2\, \mathrm{sinc}(\omega T)$$

$$\mathrm{sinc}(\theta) = \frac{\sin(\pi \theta)}{\pi \theta}$$



$$X(j\omega) = \begin{cases} 1, & -W<\omega<W \\ 0, & \text{resto} \end{cases} \longrightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega =$$
$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi} \cdot \frac{1}{jt}\left(e^{jWt} - e^{-jWt}\right) =$$
$$= \frac{1}{\pi t}\, \mathrm{sen}(Wt)$$

$$X(j\omega) = 2\pi\delta(\omega) \longrightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega = \int_{-\infty}^{+\infty} \delta(\omega)\, \underbrace{e^{j\cdot 0 \cdot t}}_{1}\, d\omega = 1$$

$$x(t) = \cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \longrightarrow X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt =$$
$$= \int_{-\infty}^{+\infty} \frac{e^{j(\omega_0-\omega)t} + e^{-j(\omega_0+\omega)t}}{2}\, dt = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

Desarrollo en Serie de Fourier (DSF)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{jkw_0 t} \quad ; \; a_k = \text{coeficientes del DSF}$$

$$x(t) = x(t+T) \quad ; \; w_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jkw_0 t}\, dt \quad \text{(Ec. análisis)}$$

↳ si $x(t)$ son sinusoides: aplicar Euler

↳ si no lo son: → Ecuación de análisis → a_k , $k \neq 0$

↳ $a_0 = \frac{1}{T}\int_T x(t)\, dt$

↳ Fórmula sumatoria de Poisson

Ej:

$$x(t) = \cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2} = \frac{1}{2} e^{jw_0 t} + \frac{1}{2} e^{-jw_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{jkw_0 t}$$

por simple inspección visual

$$k = 1 \Rightarrow a_1 = \frac{1}{2} \;,\quad k = -1 \Rightarrow a_{-1} = \frac{1}{2} \;,\quad \forall k \neq 1, -1 \Rightarrow a_k = 0$$

$$x(t) = \text{sen}(w_0 t) = \frac{1}{2j} e^{jw_0 t} - \frac{1}{2j} e^{-jw_0 t}$$

$$k = 1 \Rightarrow a_1 = \frac{1}{2j} \;,\quad k = -1 \Rightarrow a_{-1} = \frac{-1}{2j} \;,\quad \forall k \neq 1, -1 \Rightarrow a_k = 0$$

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) \cdot e^{-jkw_0 t}\, dt = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} \delta(t)\, \underbrace{\cancel{e^{-jkw_0 \cdot 0}}}_{1}\, dt = \frac{1}{T}, \forall k$$

Fórmula sumatoria de Poisson:

$$\sum_{k=-\infty}^{+\infty} x(t-kT) = \sum_{k=-\infty}^{+\infty} \underbrace{\frac{X(jk\omega_0)}{T}}_{a_k} e^{jk\omega_0 t}$$

$$\sum_{k=-\infty}^{+\infty} \delta(t-kT) = \sum \frac{1}{T} e^{jk\omega_0 t} \longrightarrow \boxed{\begin{matrix} x(t) \\ X(j\omega) \end{matrix}} \longrightarrow x(t) * \sum \delta(t-kT) = \sum x(t-kT) = \sum_k \frac{X(jk\omega_0)}{T} e^{jk\omega_0 t}$$

siendo $x(t) = x(t+T)$

$$TF(x(t)) = TF\left(\sum_k a_k e^{jk\omega_0 t}\right) = \sum_k a_k \, TF\left(e^{jk\omega_0 t}\right) =$$

$$= \sum_k 2\pi a_k \, \delta(\omega - k\omega_0)$$

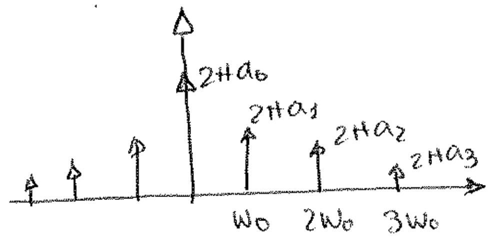


Tabla de propiedades:

$x(t) \longrightarrow X(j\omega)$

$x(t-t_0) \longrightarrow X(j\omega)\, e^{-j\omega t_0}$

$\alpha x(t) + \beta y(t) \longrightarrow \alpha X(j\omega) + \beta Y(j\omega)$

$x(t) * y(t) \longrightarrow X(j\omega) \cdot Y(j\omega)$

$x(t) \cdot y(t) \longrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$

$x(t)\, e^{j\omega_0 t} \longrightarrow X(j(\omega-\omega_0))$

$x(at) \longrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

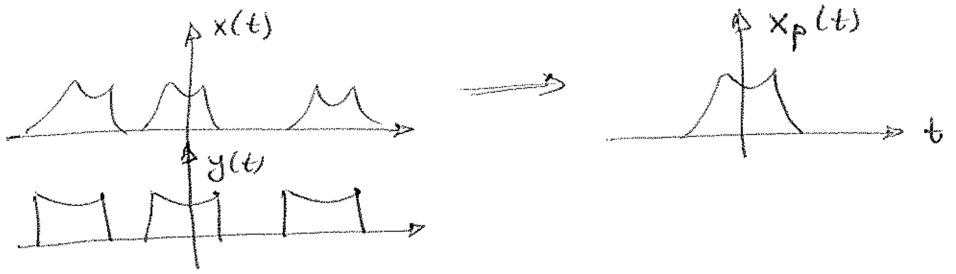
$\frac{d\,x(t)}{dt} \longrightarrow X(j\omega) \cdot j\omega$

$t \cdot x(t) \longrightarrow j \cdot \frac{d X(j\omega)}{d\omega}$

Operador convolución circular:

$$z(t) = x(t) \circledast y(t) = \int_T x(\tau)\, y(t-\tau)\, d\tau$$

↳ convolución circular



$$z(t) = \int_{-T/2}^{T/2} x_p(\tau)\, y(t-\tau)\, d\tau = x_p(t) * y(t)$$

↳ convolución lineal

$$= \int_{-\infty}^{+\infty} x_p(\tau)\, y(t-\tau)\, d\tau =$$

$$Z(j\omega) = \sum_k 2\pi\, c[k]\, \delta(\omega - k\omega_0)$$

↳ c[k] = coeficientes DSF

Análisis de sistemas LTI:

↳ señales NO periódicas:

$$x(t),\ X(j\omega) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(j\omega) \end{matrix}} \longrightarrow y(t) = x(t) * h(t),\quad Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

↳ señales SI periódicas:

$$x(t) = x(t+T) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(j\omega) \end{matrix}} \longrightarrow y(t) = x(t) * h(t) = \sum y_p(t-kT)$$

$$x(t) = \sum a_k e^{jk\omega_0 t} \qquad y(t) = \sum b_k e^{jk\omega_0 t}$$

$$X(j\omega) = \sum 2\pi a_k \delta(\omega - k\omega_0) \qquad Y(j\omega) = \sum 2\pi\, a_k H(j\omega)\, \delta(\omega - k\omega_0)$$

$$= \sum 2\pi \underbrace{a_k\, H(jk\omega_0)}_{b_k} \delta(\omega - k\omega_0)$$

$$x(t) = \sum x_p(t-kT) \qquad y(t) = \sum y_p(t-kT)$$

Ej 4.21:

a) $x(t) = e^{-at} \cos(w_0 t)\, u(t)$, $a > 0$

1st way: $X(jw) = \int_{-\infty}^{+\infty} x(t)\, e^{-jwt} dt = \int_{0}^{+\infty} e^{-at} \left(\frac{1}{2} e^{jw_0 t} + \frac{1}{2} e^{-jw_0 t}\right) e^{-jwt} dt$

$$= \int_0^{+\infty} \frac{1}{2} e^{-(a+j(w-w_0))t} dt + \frac{1}{2} \int_0^{\infty} e^{-(a+j(w+w_0))t} dt$$

$$= \frac{1/2}{a+j(w-w_0)} + \frac{1/2}{a+j(w+w_0)}$$

2nd way: $x_1(t) = e^{-at} \cdot u(t) \longleftrightarrow X_1(jw) = \frac{1}{a+jw}$

$x_2(t) = \cos(w_0 t) \longleftrightarrow X_2(jw) = \pi\delta(w-w_0) + \pi\delta(w+w_0)$

$x(t) = x_1(t) \cdot x_2(t) \longleftrightarrow X(jw) = \frac{1}{2\pi} X_1(jw) * X_2(jw)$

$$X(jw) = \frac{1}{2} X_1(j(w-w_0)) + \frac{1}{2} X_1(j(w+w_0)) =$$

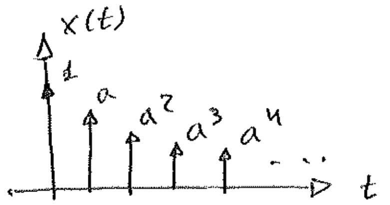
$$= \frac{1/2}{a+j(w-w_0)} + \frac{1/2}{a+j(w+w_0)}$$

3rd way: $x(t) = \frac{1}{2} \underbrace{e^{-at} u(t)}_{x_1(t)} e^{jw_0 t} + \frac{1}{2} \underbrace{e^{-at} u(t)}_{x_1(t)} e^{-jw_0 t}$

$$X(jw) = \frac{1}{2} X_1(j(w-w_0)) + \frac{1}{2} X_1(j(w+w_0))$$

Ej 4.21:

d) $x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT) \qquad a \in (0,1)$



$$X(jw) = \sum_{k=0}^{\infty} a^k \, TF(\delta(t-kT))$$

$$= \sum_{k=0}^{\infty} a^k e^{-jwkT} = \sum_{k=0}^{\infty} \left(a \cdot e^{-jwT}\right)^k$$

$$= \frac{1}{1 - a e^{-jwT}}$$

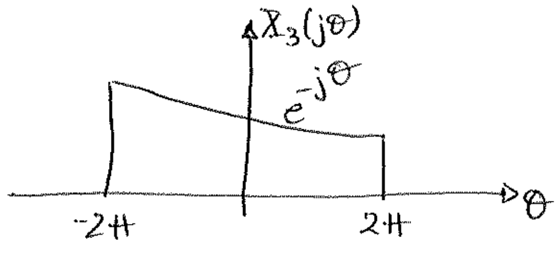
d) $x(t) = \dfrac{\operatorname{sen}(\pi t)}{\pi t} \cdot \dfrac{\operatorname{sen}(2\pi(t-1))}{\pi(t-1)}$

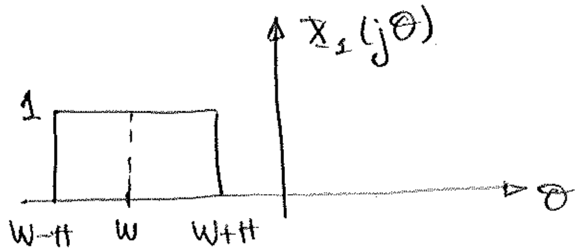
$$x_1(t) = \frac{\operatorname{sen}(\pi t)}{\pi t} \longleftrightarrow X_1(jw) = \begin{cases} 1 & |w| < \pi \\ 0 & \text{resto} \end{cases}$$

$$x_2(t) = \frac{\operatorname{sen}(2\pi t)}{\pi t} \longleftrightarrow X_2(jw) = \begin{cases} 1 & |w| < 2\pi \\ 0 & \text{resto} \end{cases}$$

$$x_3(t) = x_2(t-1) \longleftrightarrow X_3(jw) = X_2(jw) \cdot e^{-jw}$$

$$X(jw) = \frac{1}{2\pi} X_1(jw) * X_3(jw) = \frac{1}{2\pi} \int X_3(j\theta) X_1(j(w-\theta)) \, d\theta$$





$w + \pi < -2\pi : X(jw) = 0$

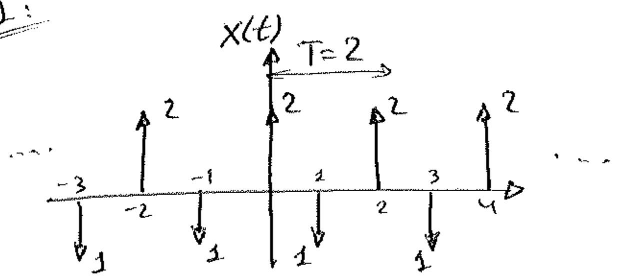
$-2\pi < w + \pi < 0 : \quad X(jw) \frac{1}{2\pi} \int_{-2\pi}^{w+\pi} e^{-j\theta} \, d\theta = \frac{1}{2\pi} \cdot \frac{1}{-j} \left(e^{-j(w+\pi)} - e^{j2\pi} \right)$

$0 < w + \pi < 2\pi : \quad X(jw) = \frac{1}{2\pi} \int_{w-\pi}^{w+\pi} e^{-j\theta} \, d\theta$

$0 < w - \pi < 2\pi : \quad X(jw) = \frac{1}{2\pi} \int_{w-\pi}^{2\pi} e^{-j\theta} \, d\theta$

$w - \pi > 2\pi \quad : X(jw) = 0$

Ej 4.21: hj



1st way:

$$x_1(t) = \sum_{k=-\infty}^{+\infty} \delta(t - k\cdot 2) \longleftrightarrow X_1(j\omega) = \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - k\pi)$$

$$x(t) = 2\cdot x_1(t) - x_1(t-1)$$

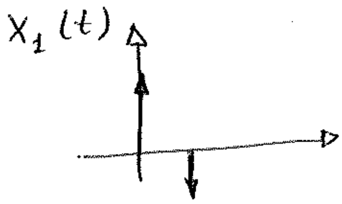
$$X(j\omega) = 2\, X_1(j\omega) - X_1(j\omega)\, e^{-j\omega} = \left(2 - e^{-j\omega}\right) X_1(j\omega) =$$

$$= \left(2 - e^{-j\omega}\right) \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - k\pi) = \sum_{k} \pi \left(2 - e^{-j\omega}\right)\Big|_{\omega = k\pi} \delta(\omega - k\pi)$$

$$= \sum_{k=-\infty}^{+\infty} \pi \left(2 - e^{-jk\pi}\right) \delta(\omega - k\pi)$$

2nd way:

$$X(j\omega) = \sum_{k} 2\pi\, a_k\, \delta(\omega - k\pi)$$



$$\longrightarrow X_1(j\omega) = 2 - e^{-j\omega}$$

$$a_k = \frac{X_1(j\omega)\big|_{\omega = k\pi}}{2} = \frac{2 - e^{-jk\pi}}{2}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \pi \left(2 - e^{-jk\pi}\right) \delta(\omega - k\pi)$$

Ej 4.22:

a) $$X(j\omega) = \frac{2\,\mathrm{sen}(3(\omega-2\pi))}{\omega - 2\pi}$$

$$X_1(j\omega) = \frac{2\,\mathrm{sen}(3\omega)}{\omega} \longleftrightarrow x_1(t) = \begin{cases} 1, & |t|<3 \\ 0, & \text{resto} \end{cases}$$

$$X(j\omega) = X_1(j(\omega-2\pi)) \longleftrightarrow x(t) = x_1(t)\, e^{j2\pi t} = \begin{cases} e^{j2\pi t}, & |t|<3 \\ 0, & \text{resto} \end{cases}$$

b) $$X(j\omega) = \cos(4\omega + \pi/3)$$

$$X(j\omega) = \frac{e^{j\pi/3}\, e^{j4\omega}}{2} + \frac{e^{-j\pi/3}\, e^{-j4\omega}}{2}$$

$$x(t) = \frac{e^{j\pi/3}}{2}\, TF^{-1}(e^{j4\omega}) + \frac{e^{-j\pi/3}}{2}\, TF^{-1}(e^{-j4\omega})$$

$$= \frac{e^{j\pi/3}}{2}\,\delta(t+4) + \frac{e^{-j\pi/3}}{2}\,\delta(t-4)$$

Ej 4.32:

$$x(t) \longrightarrow \boxed{h(t) = \frac{\mathrm{sen}(4(t-1))}{\pi(t-1)}} \longrightarrow y(t)$$

$$h_1(t) = \frac{\mathrm{sen}(4t)}{\pi \cdot t} \longleftrightarrow H_1(j\omega) = \begin{cases} 1, & |\omega|<4 \\ 0, & \text{resto} \end{cases}$$

$$h(t) = h_1(t-1) \longleftrightarrow H(j\omega) = H_1(j\omega)\, e^{-j\omega} = \begin{cases} e^{-j\omega}, & |\omega|<4 \\ 0, & \text{resto} \end{cases}$$

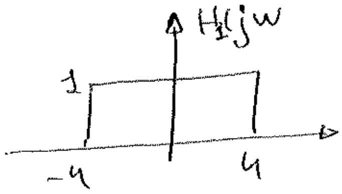
$$h(t) = h_1(t) * \delta(t-1)$$

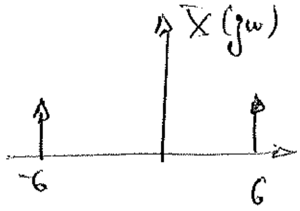
$$H(j\omega) = H_1(j\omega)\cdot e^{-j\omega}$$

$$x(t) \longrightarrow \boxed{\begin{array}{c} h_1(t) = \dfrac{\mathrm{sen}(4t)}{\pi t} \\ H_1(j\omega) = \begin{cases} 1, & |\omega|<4 \\ 0 & \text{resto} \end{cases} \end{array}} \xrightarrow{y(t)} \boxed{\begin{array}{c} h_2(t) = \delta(t-1) \\ H_2(j\omega) = e^{-j\omega} \end{array}} \longrightarrow z(t)$$

b) $x(t) = \cos\left(6t + \frac{\pi}{2}\right) = \frac{1}{2} e^{j\pi/2} e^{j6t} + \frac{1}{2} e^{-j\pi/2} e^{-j6t}$

$X(j\omega) = \pi\, e^{j\pi/2}\, \delta(\omega - 6) + \pi\, e^{-j\pi/2}\, \delta(\omega + 6)$



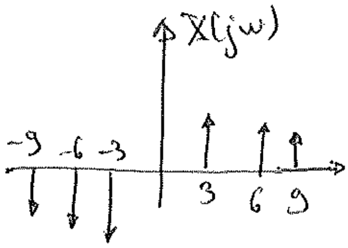


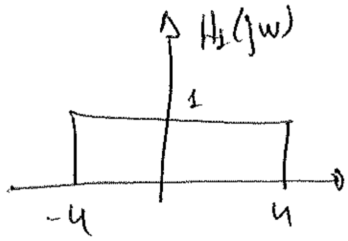
$\Rightarrow$ $Y(j\omega) = 0$

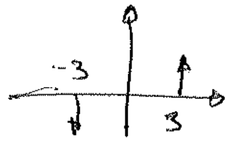
$y(t) = 0$

$z(t) = 0$

c) $x(t) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \operatorname{sen}(3kt)$





$\Rightarrow$ $Y(j\omega) =$ 

$y(t) = \frac{1}{2} \cdot \operatorname{sen}(3 \cdot t)$

$(k = 1)$

$z(t) = \frac{1}{2} \operatorname{sen}(3(t-1))$

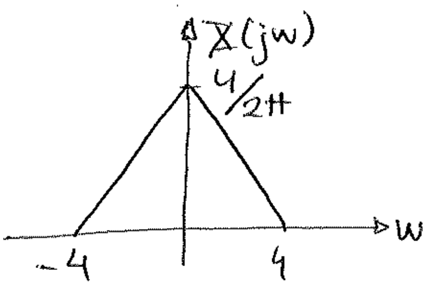
d) $x(t) = \dfrac{\operatorname{sen}(4(t+1))}{\pi(t+1)} = \dfrac{\operatorname{sen}(4t)}{\pi t} * \delta(t+1)$

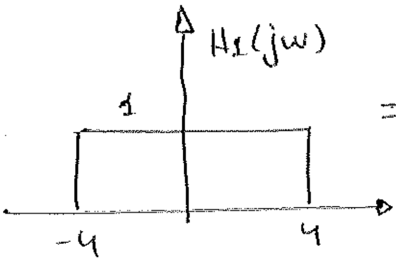
$$X(j\omega) = \begin{Bmatrix} 1, & |\omega| < 4 \\ 0, & \text{resto} \end{Bmatrix} \cdot e^{j\omega}$$

$y(t) = x(t) \longrightarrow z(t) = \left(\dfrac{\sin(4t)}{\pi t} * \delta(t+1)\right) * \delta(t-1) = \dfrac{\operatorname{sen}(4t)}{\pi t}$

e) $x(t) = \left(\dfrac{\operatorname{sen}(2t)}{\pi t}\right)^2 = x_1(t) \cdot x_1(t)$; $x_1(t) = \dfrac{\operatorname{sen}(2t)}{\pi t}$

$X(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_1(j\omega)$; $X_1(j\omega) = \begin{cases} 1, & |\omega| < 2 \\ 0, & \text{resto} \end{cases}$





$\Rightarrow$ $Y(j\omega) = X(j\omega)$

$y(t) = x(t)$

$z(t) = y(t-1)$

$z(t) = \left(\dfrac{\operatorname{sen}(2(t-1))}{\pi(t-1)}\right)^2$

Enventanado:

Enventanar es multiplicar en el dominio del tiempo una señal por otra limitada en anchura.

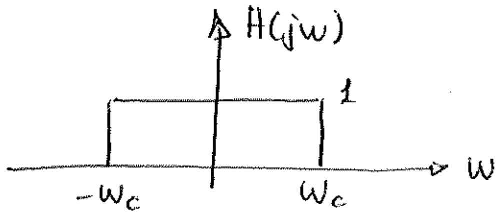
$$y(t) = x(t) \cdot v(t) \leftrightarrow Y(jw) = \frac{1}{2\pi} X(jw) * V(jw)$$

Filtrado:

señales que discriminan bandas de frecuencias

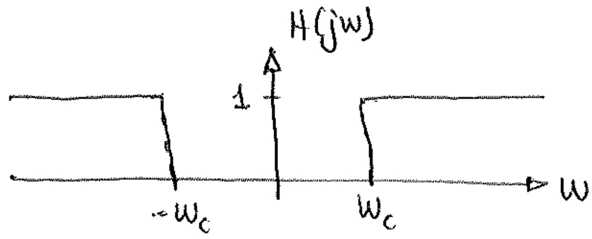
→ Filtro paso bajo:

$$H(jw) = \begin{cases} 1 & |w| < w_c \\ 0 & \text{resto} \end{cases}$$



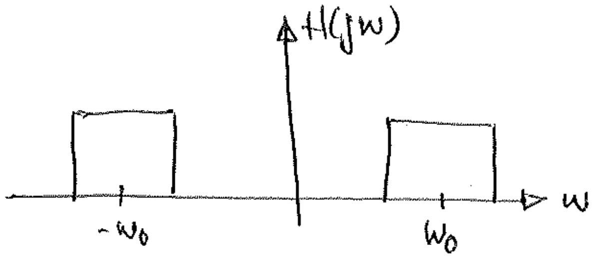
→ Filtro paso alto:

$$H(jw) = \begin{cases} 0 & |w| < w_c \\ 1 & \text{resto} \end{cases}$$



→ Filtro paso banda:

$$H(jw) = \begin{cases} 1, & w_1 < |w| < w_2 \\ 0, & \text{resto} \end{cases}$$



Transformada bilateral de Laplace

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

Ec. de análisis — Ec. de síntesis

$$e^{j\omega_0 t} \longrightarrow y(t) = x(t) * h(t) = \int x(t-\tau)\, h(\tau)\, d\tau = \int e^{j\omega_0 (t-\tau)}\, h(\tau)\, d\tau$$

Fourier

$$= e^{j\omega_0 t} \int e^{-j\omega_0 \tau}\, h(\tau)\, d\tau = e^{j\omega_0 t} H(j\omega_0)$$

$$e^{\sigma_0 + j\omega_0 t} \longrightarrow y(t) = x(t) * h(t) = \int x(t-\tau)\, h(\tau)\, d\tau = \int e^{s_0 (t-\tau)}\, h(\tau)\, d\tau$$

Laplace

$$= e^{s_0 t} \int e^{-s_0 \tau}\, h(\tau)\, d\tau = e^{s_0 t} H(s_0)$$

$X(s)$ es una función compleja de variable compleja
Además de su expresión, deberemos indicar la región del plano donde dicha expresión existe (ROC):

- Las fronteras son líneas verticales
- si $x(t)$ es de longitud finita y absolutamente integrable, ROC = todo el plano
- si $x(t)$ es infinita a derechas, la ROC = semiplano a derechas
- si $x(t)$ es infinita a izquierdas, ROC = semiplano a izq
- si $x(t)$ es infinita a ambos lados, la ROC (si existe) será una franja vertical.

Algunas propiedades:

$$x(t) = x_1(t) + x_2(t) \longleftrightarrow X(s) = X_1(s) + X_2(s)$$
$$R \supset R_1 \cap R_2$$

$$x(t) = x_1(t-t_0) \longleftrightarrow X(s) = X_1(s)\, e^{-st_0} \quad , \quad R = R_1$$

$$x(t) = x_1(t) * x_2(t) \longleftrightarrow X(s) = X_1(s) \cdot X_2(s)$$
$$R \supset R_1 \cap R_2$$

$$x(t) = \frac{d\, x_1(t)}{dt} \longleftrightarrow X(s) = s \cdot X_1(s) \quad , \quad R \supset R_1$$

$$x(t) = t \cdot x_1(t) \longleftrightarrow X(s) = -\frac{dX_1(s)}{ds} \quad , \quad R = R_1$$

Ej: $X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$, ROC

$$x(t) = \delta(t) \longleftrightarrow X(s) = 1 \ , \ \forall s$$
$$x(t) = \delta'(t) \longleftrightarrow X(s) = s \ , \ \forall s$$
$$x(t) = \delta^{(n}(t) \longleftrightarrow X(s) = s^n \ , \ \forall s$$

$$x(t) = e^{-at}u(t) \longleftrightarrow X(s) = \int_{-\infty}^{+\infty} e^{-at}u(t)\, e^{-st}\, dt =$$
$$= \int_0^{\infty} e^{-(a+s)t}\, dt = \frac{1}{-(a+s)}\, e^{-(a+s)t}\Big|_0^{\infty} =$$
$$= \frac{e^{-(a+\sigma+j\omega)t}}{a+s}\Bigg|_{\infty}^{0} = \frac{1}{a+s} \ , \quad \begin{array}{l} a+\sigma > 0 \\ \sigma > -a \end{array}$$

si $a = 0$

$$x(t) = u(t) \longleftrightarrow X(s) = \frac{1}{s} \ , \ \sigma > 0 \ // \ X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$x(t) = -e^{-at}u(-t) \longleftrightarrow X(s) = \int_{-\infty}^{0} e^{-at} e^{-st}\, dt = \frac{1}{a+s}\, e^{-(a+\sigma+j\omega)}\Big|_{-\infty}^{0}$$
$$= \frac{1}{a+s} \ , \quad \begin{array}{l} a+\sigma < 0 \\ \sigma < -a \end{array}$$

$$\frac{1}{s+a} \begin{cases} \xrightarrow{\sigma > -a} & e^{-at} u(t) \\ \xrightarrow[\sigma < -a]{} & -e^{-at} u(-t) \end{cases}$$

$$x(t) = t \cdot e^{-at} u(t)$$

$$x_1(t) = e^{-at} u(t) \longleftrightarrow X_1(s) = \frac{1}{a+s}$$

$$x(t) = t \cdot x_1(t) \longleftrightarrow X(s) = \frac{1}{(a+s)^2}$$

$$x(t) = \frac{t^2}{2} e^{-at} u(t) = \frac{t}{2} (t e^{-at} u(t))$$

$$x_1(t) = t e^{-at} u(t) \longleftrightarrow X_1(s) = \frac{1}{(a+s)^2}$$

$$x(t) = t \cdot x_1(t) \longleftrightarrow X(s) = \frac{1}{(a+s)^3}$$

$$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \longleftrightarrow \frac{1}{(a+s)^n}, \quad \sigma > -a$$

$$\frac{-t^{n-1}}{(n-1)!} e^{-at} u(-t) \longleftrightarrow \frac{1}{(a+s)^n}, \quad \sigma < -a$$

Sistemas en Ecuaciones Diferenciales

$$\underbrace{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}}_{H(s) = \frac{Y(s)}{X(s)}} \quad ; \text{ CRI} \Longrightarrow \text{Sistema LTI}$$

polos: raíces del denominador de $H(s)$

ceros: raíces del numerador de $H(s)$

$H(jw)$ existe y vale $H(s)|_{s=jw}$ sólo si $jw \in ROC$

$$\sum_k a_k \frac{d^k y(t)}{dt^k} = \sum_k b_k \frac{d^k x(t)}{dt^k}$$

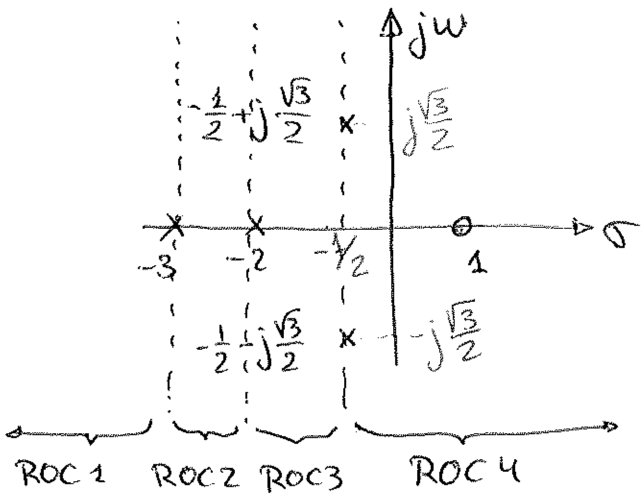
$$\sum_k a_k s^k Y(s) = \sum_k b_k s^k X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_k b_k s^k}{\sum_k a_k s^k} = A \frac{\prod_i (s - s_{ci})}{\prod_i (s - s_{pi})}$$

s_{ci} = ceros $_i$
s_{pi} = polos $_i$

Ej 9.7: $H(s) = \dfrac{s-1}{(s+2)(s+3)(s^2+s+1)}$

$s-1=0 \Leftrightarrow s=1$ ← cero
$s+2=0 \Leftrightarrow s=-2$
$s+3=0 \Leftrightarrow s=-3$
$s^2+s+1=0 \Leftrightarrow s=-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$
} polos



los ceros se representan con "o"
los polos se representan con "x"

los polos marcan las fronteras de las ROCs

ROC 1: $\sigma < -3$
ROC 2: $-3 < \sigma < -2$
ROC 3: $-2 < \sigma < -1/2$
ROC 4: $-1/2 < \sigma$

sólo existe $H(j\omega) = H(s)|_{j\omega}$ sólo en ROC 4

$$H(s) = \frac{\text{Numerador}(s)}{\text{Denominador}(s)} = \frac{N(s)}{D(s)}$$

Si Grado $(N(s)) <$ Grado $(D(s)) \Rightarrow$ fracciones simples

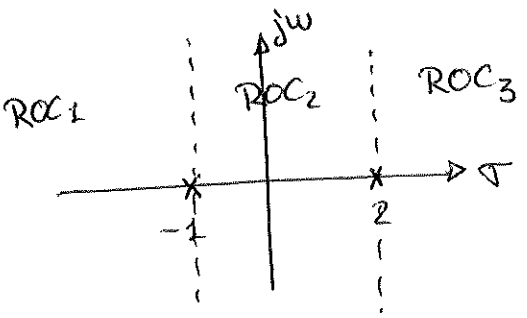
Si Grado $(N(s)) \geq$ Grado $(D(s)) \Rightarrow$ división de polinomios

Ej: 9.31:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2\,y(t) = x(t)$$

$$\to s^2\,Y(s) - s\,Y(s) - 2\,Y(s) = X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2-s-2} = H(s) = \frac{1}{(s+1)(s-2)}$$



$\nexists\, H(jw)$ en ROC 1 ni ROC 3

en $ROC_2 \Rightarrow H(jw) = H(s)\big|_{s=jw}$

$$H(jw) = \frac{1}{(jw+1)(jw-2)}$$

Para hallar h_i, como $G(1) < G((s+1)(s-2))$: fracc. simples

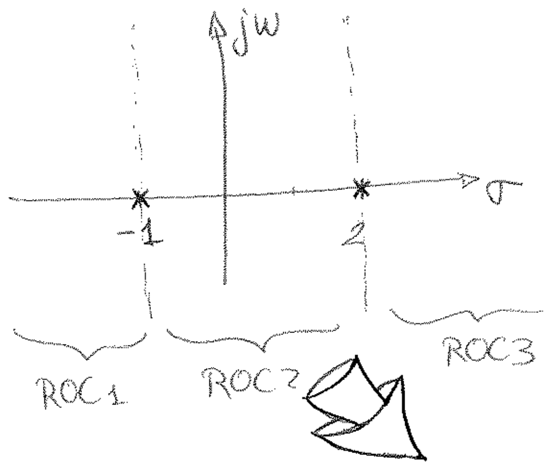
$$H(s) = \frac{A}{s+1} + \frac{B}{s-2}$$

$$\frac{A}{s+1} \xrightarrow{\sigma > -1} A\,e^{-t}u(t)$$

$$\frac{A}{s+1} \xrightarrow{\sigma < -1} -A\,e^{-t}u(-t)$$

$$\frac{B}{s-2} \xrightarrow{\sigma > 2} B\,e^{2t}u(t)$$

$$\frac{B}{s-2} \xrightarrow{\sigma < 2} -B\,e^{2t}u(-t)$$



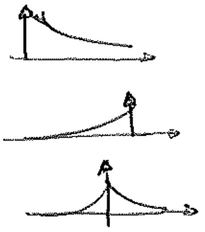
$ROC_1 \Rightarrow h_1(t) = -A e^{-t} u(-t) - B e^{2t} u(-t)$

$ROC_2 \Rightarrow h_2(t) = A e^{-t} u(t) - B e^{2t} u(-t)$

$ROC_3 \Rightarrow h_3(t) = A e^{-t} u(t) + B e^{2t} u(t)$

Clasificación:

- causal
- anticausal
- no causal



$ROC_1 \Rightarrow h_1(t) =$ anticausal

$ROC_2 \Rightarrow h_2(t) =$ no causal

$ROC_3 \Rightarrow h_3(t) =$ causal

estable $\int |h(t)|\,dt < \infty \Leftrightarrow \sigma = 0$ $\quad ROC_2 \Rightarrow h_2(t) =$ estable

$\Updownarrow$

ROC $\subset$ eje jw

$$H(s) = \frac{A}{s+1} + \frac{B}{s-2}$$

$$\hookrightarrow H(s)\cdot(s+1) = A + \left.\frac{B(s+1)}{(s-2)}\right|_{s=-1} = A = H(s)(s+1)\Big|_{s=-1} = \frac{-1}{3}$$

$$\hookrightarrow H(s)\cdot(s-2) = \left.\frac{A(s-2)}{s+1} + B\right|_{s=2} = B = H(s)\cdot(s-2)\Big|_{s=2} = \frac{1}{3}$$

Ej: $H(s) = \dfrac{1}{(s+1)^2\cdot(s-2)}$

$$H(s) = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s-2}$$

$\frac{A}{(s+1)^2}$: $\sigma < -1 \rightarrow -Ate^{-t}u(-t)$; $\sigma > -1 \rightarrow Ate^{-t}u(t)$

$\frac{B}{s+1}$: $\sigma < -1 \rightarrow -Be^{-t}u(-t)$; $\sigma > -1 \rightarrow Be^{-t}u(t)$

$\frac{C}{s-2}$: $\sigma < 2 \rightarrow -Ce^{-t}u(-t)$; $\sigma > 2 \rightarrow Ce^{-t}u(t)$

$\frac{1}{s+a}$: $\sigma > -a \rightarrow e^{-at}u(t)$; $\sigma < -a \rightarrow -e^{-at}u(-t)$

$\frac{1}{(s+a)^2}$: $\sigma > -a \rightarrow te^{-at}u(t)$; $\sigma < -a \rightarrow -te^{-at}u(-t)$

$h_1(t) = -At\,e^{-t}u(-t) - B\,e^{-t}u(-t) - C\,e^{-t}u(-t)$ $\left\{ \begin{array}{l} \text{anticausal} \\ \text{no estable} \end{array} \right.$

$h_2(t) = At\,e^{-t}u(t) + B\,e^{-t}u(t) - C\,e^{-t}u(-t)$ $\left\{ \begin{array}{l} \text{no causal} \\ \text{estable} \end{array} \right.$

$h_3(t) = At\,e^{-t}u(t) + B\,e^{-t}u(t) + C\,e^{-t}u(t)$ $\left\{ \begin{array}{l} \text{causal} \\ \text{no estable} \end{array} \right.$

$$\hookrightarrow H(s)(s-2) = \frac{1}{(s+1)^2} = \frac{A(s-2)}{(s+1)^2} + \frac{B(s-2)}{s+1} + C \Bigg|_{s=2} \Rightarrow C = H(s)\cdot(s-2) = \frac{1}{9}$$

$$\hookrightarrow H(s)(s+1)^2 = \underbrace{\frac{1}{s-2} = A + B(s+1) + C\frac{(s+1)^2}{s-2}}_{\text{derivamos:}} \Bigg|_{s=-1} \Rightarrow A = H(s)(s+1)^2\Big|_{s=-1}$$

$$\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{-1}{(s-2)^2} = 0 + B + C\cdot\frac{2(s+1)(s-2)-(s+1)^2}{(s-2)^2}\Bigg|_{s=-1} \Rightarrow$$

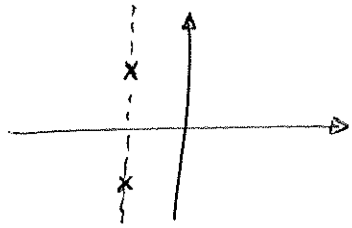
$$\Rightarrow B = \frac{d}{ds}\left(H(s)\cdot(s+1)^2\right)\Bigg|_{s=-1}$$

Ej: $H(s) = \dfrac{1}{s^2+2s+5}$, $\sigma > -1$

$\hookrightarrow s = -1 \pm j2$

$\hookrightarrow S_1 = \sigma_0 + jw_0$

$\hookrightarrow S_2 = \sigma_0 - jw_0$



$$H(s) = \frac{1}{(s-s_1)(s-s_2)} = \frac{A}{s-s_1} + \frac{B}{s-s_2}$$

$$A = H(s)\cdot(s-s_1) = A + \frac{B(s-s_1)}{s-s_2}\Bigg|_{s=s_1} \Rightarrow A = H(s)\cdot(s-s_1)\Big|_{s=s_1} = \frac{1}{2jw_0}$$

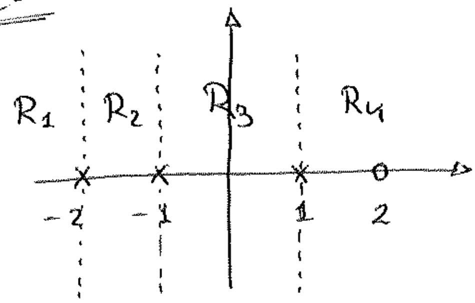
$$B = H(s)\cdot(s-s_2) = \frac{A(s-s_2)}{s-s_1} + B\Bigg|_{s=s_2} \Rightarrow B = H(s)(s-s_2)\Big|_{s=s_2} = \frac{1}{-2jw_0}$$

$B = A^*$

si

$S_2 = S_1^*$

Ej 9.27:



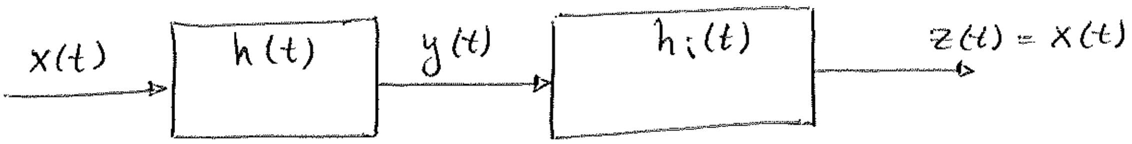
R_1: anticausal, no estable

R_2: no causal, no estable

R_3: no causal, estable

R_4: causal, no estable

Sistema inverso:



$$h(t) * h_i(t) = \delta(t)$$

$$H(j\omega) \cdot H_i(j\omega) = 1 \rightarrow H_i(j\omega) = \frac{1}{H(j\omega)}$$

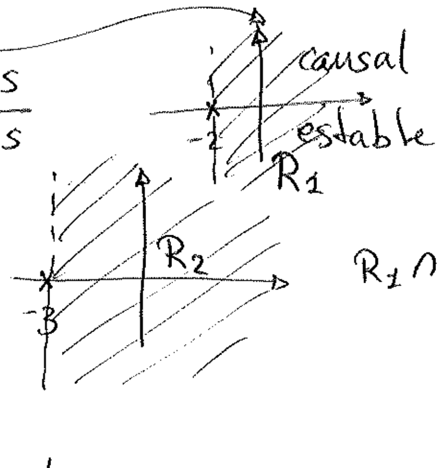
$$\underbrace{H(s)}_{R_1} \cdot \underbrace{H_i(s)}_{R_2} = 1 \rightarrow H_i(s) = \frac{1}{H(s)}$$

$$R_1 \cap R_2 \neq \emptyset$$

Ej: $h(t) = \delta(t) + e^{-2t} u(t)$

$$H(s) = 1 + \frac{1}{2+s} = \frac{3+s}{2+s}$$

$$H_i(s) = \frac{1}{H(s)} = \frac{2+s}{3+s}$$



$R_1 \cap R_2 \neq \emptyset$

Grado (Num) = Grado (Denom)

$\Downarrow$

$$\frac{2+s}{3+s} = \{\text{dividiendo}\} = 1 - \frac{1}{s+3} = H_i(s)$$

$$h_i(t) = \delta(t) - e^{-3t} u(t)$$

Ej: 4.33

LTI causal

$$x(t) \rightarrow \boxed{\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)} \rightarrow y(t)$$

$$s^2 Y(s) + 6s\,Y(s) + 8\,Y(s) = 2\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{s^2+6s+8} = \frac{2}{(s+2)(s+4)}$$

$$H(s) = \frac{A}{s+2} + \frac{B}{s+4}$$

$$A = H(s)\cdot(s+2)\Big|_{s=-2} = \frac{2}{s+4}\Big|_{s=-2} = 1$$

$$B = H(s)\cdot(s+4)\Big|_{s=-4} = \frac{2}{s+2}\Big|_{s=-4} = -1$$

$$h(t)_{\text{causal}} = A e^{-2t}u(t) + B e^{-4t}u(t)$$

siendo $x(t) = t\,e^{-2t}u(t)$ ... $y(t)$??

$$y(t) = x(t) * h(t) \quad :($$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}(X(s)\cdot H(s)) \; ; \; X(s) = \frac{1}{(2+s)^2}, \; \sigma > -2$$

$$Y(s) = \frac{2}{(2+s)^2}\cdot\frac{1}{(s+2)(s+4)} = \frac{2}{(s+2)^3(s+4)}$$

$$Y(s) = \frac{A}{(s+2)^3} + \frac{B}{(s+2)^2} + \frac{C}{s+2} + \frac{D}{s+4}$$

$$y(t) = A\frac{t^2}{2}e^{-2t}u(t) + B\,t\,e^{-2t}u(t) + C\,e^{-2t}u(t) + D\,e^{-4t}u(t)$$

sea ahora $x'(t) = (t-t_0)\,e^{-2(t-t_0)}\,u(t-t_0)$

$y'(t) = y(t-t_0)$ (aprovechando propiedad invarianza temporal)

Transformada Laplace Unilateral:

útil para analizar ecuaciones diferenciales con condiciones iniciales no nulas

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt \qquad x(t) = \begin{cases} x(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

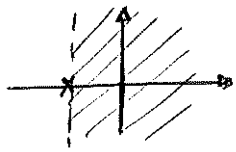
condiciones de reposo inicial: $\frac{d\,x(t)}{dt} \longleftrightarrow s \cdot X(s)$

condiciones de no reposo inicial: $\frac{d\,x(t)}{dt} \longleftrightarrow s\,X(s) - x(0)$

Ej: $\frac{dy(t)}{dt} + 5y(t) = x(t)$; $x(t) = 3e^{-2t} \cdot u(t)$

a) bilateral, causal:

$$sY(s) + 5Y(s) = X(s) \Rightarrow H(s) = \frac{1}{s+5} \; ;$$



$$h(t) = e^{-5t} \cdot u(t)$$

b) unilateral con CRI:
exactamente lo mismo que a)

c) unilateral siendo $y(0) = -2$

a) b)

$$Y(s) = H_{CRI}(s) \cdot X(s) = \frac{1}{s+5} \cdot \frac{1}{s+2} = \frac{A}{s+5} + \frac{B}{s+2}$$

$$A = Y(s) \cdot (s+5)\Big|_{s=-5} = \frac{3}{s+2}\Big|_{s=-5} = -1 \; ; \; B = 1$$

$$y_{CRI}(t) = A \cdot e^{-5t} u(t) + B\, e^{-2t} u(t)$$

c) $sY(s) - (-2) + 5Y(s) = X(s)$

$$Y(s) = X(s) \cdot \frac{1}{s+5} - 2\, \frac{1}{s+5}$$

$$y(t) = y_{CRI}(t) - 2 \cdot e^{-5t} u(t)$$

Señales paso banda y paso bajo equivalente

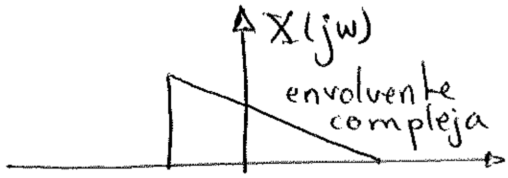
una señal modulada linealmente responde genéricamente

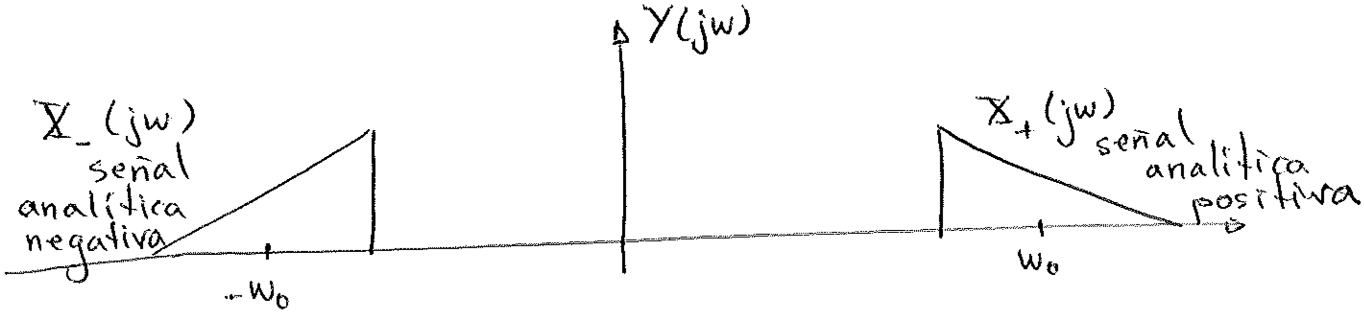
$$y(t) = A\cos(w_0 t + \Theta(t)) \quad ; \quad A = a(t)$$

$$y(t) = \underbrace{\frac{1}{2}\underbrace{a(t)\, e^{j\Theta(t)}}_{x(t)} \cdot e^{jw_0 t}}_{\substack{X_+(t) \\ \text{señal analítica positiva}}} + \underbrace{\frac{1}{2}\underbrace{a(t)\, e^{-j\Theta(t)}}_{x^*(t)}\, e^{-jw_0 t}}_{\substack{X_-(t) = X_+^*(t) \\ \text{señal analítiga negativa}}}$$

$x(t)$ = envolvente compleja

$$Y(jw) = \frac{1}{2} X(j(w-w_0)) + \frac{1}{2} X^*(-j(w-w_0))$$





Transformador de Hilbert:

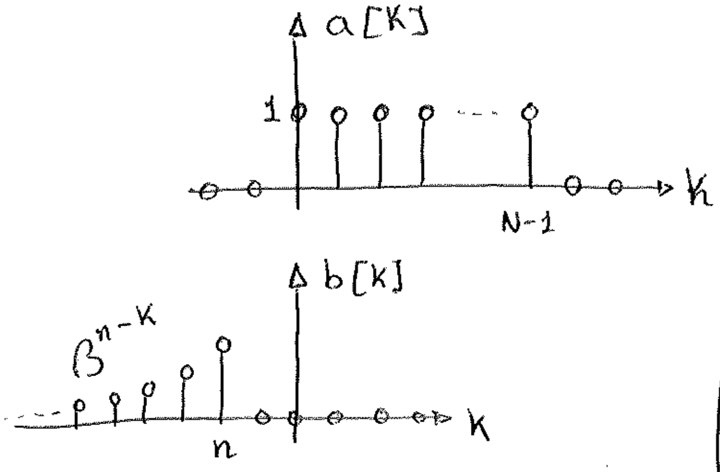
$$h(t) = \frac{1}{\pi t} \longleftrightarrow H(jw) = -j\,\mathrm{sign}(w)$$

Feb 2010 P1

a) $a[n] = u[n] - u[n-N]$

$b[n] = \beta^n u[n] \quad \beta \in (0,1)$

$$c[n] = a[n] * b[n] = \sum_{k=-\infty}^{+\infty} a[k] \cdot b[n-k]$$



$n<0: \quad c[n] = 0$

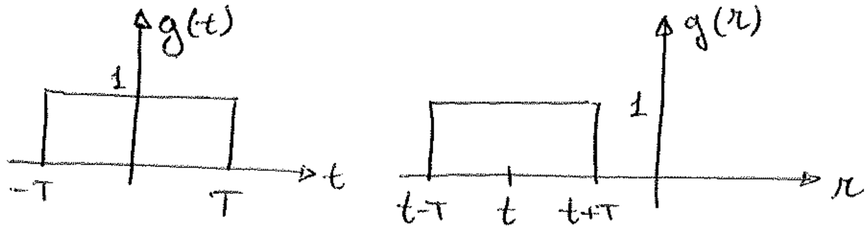
$$0 \leq n \leq N-1: \quad c[n] = \sum_{k=0}^{n} \beta^{n-k} = \beta^n \cdot \frac{1-\beta^{-(n+1)}}{1-\beta^{-1}}$$

$$n > N-1: \quad c[n] = \sum_{k=0}^{N-1} \beta^{n-k} = \beta^n \cdot \frac{1-\beta^{-N}}{1-\beta^{-1}}$$

b) $f(t) = \cos(w_0 t)$

$g(t) = u(t+T) - u(t-T)$

$$m(t) = f(t) * g(t) = \int f(\tau)\, g(t-\tau)\, d\tau$$



$$m(t) = \int_{t-T}^{t+T} \cos(w_0 \tau)\, d\tau = \frac{1}{w_0} \operatorname{sen}(w_0 \tau) \Big|_{t-T}^{t+T}$$

c) $x(t) = \sum_n a[n]\, f(t-nT) \quad ; \quad y(t) = \sum_k b[k]\, g(t-kT)$

$z(t) = x(t) * y(t) = f(c[n], m(t))$

$c[n] = a[n] * b[n]$

$m(t) = f(t) * g(t)$

$$z(t) = \left[f(t) * \sum_n a[n]\,\delta[t-nT] \right] * \left[g(t) * \sum_k b[k]\,\delta(t-kT) \right] =$$

$$= \Big(f(t) * g(t)\Big) * \left(\sum_n a[n]\,\delta(t-nT) * \sum_k b[k]\,\delta(t-kT) \right) =$$

$$= m(t) * \sum_n \sum_k a[n]\,b[k]\;\delta(t-(n+k)T) \qquad \Big(\big(n+k=\ell\big)\Big)$$

$$= m(t) * \sum_\ell \left(\sum_k a[\ell-k]\cdot b[k] \right) \delta(t-\ell T) =$$

$$= m(t) * \sum_\ell c[\ell]\;\delta(t-\ell T) = \sum_\ell c[\ell]\cdot m(t-\ell T)$$

Feb 2002 P1

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{Lineal?}\\ \text{Invar?}\\ h(t)?\end{matrix}} \longrightarrow y(t) = \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} x(\tau)\,d\tau$$

a) Linealidad:

$$x_1(t) \longrightarrow y_1(t) = \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} x_1(\tau)\,d\tau$$

$$x_2(t) \longrightarrow y_2(t) = \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} x_2(\tau)\,d\tau$$

$$\alpha x_1(t) + \beta x_2(t) = x_3(t) \longrightarrow y_3(t) \overset{??}{=} \alpha y_1(t) + \beta y_2(t)$$

$$y_3(t) = \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} \left[\alpha x_1(\tau) + \beta x_2(\tau)\right] d\tau =$$

$$= \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} \alpha x_1(\tau)\,d\tau + \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} \beta x_2(\tau)\,d\tau = \alpha y_1(t) + \beta y_2(t)$$

sí cumple $\Downarrow$ Lineal

b) Invarianza:

$$x_0(t) \longrightarrow y_0(t) = \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} x_0(\tau)\,d\tau$$

$$x_1(t) = x_0(t-t_0) \longrightarrow y_1(t) = \frac{1}{T_1+T_2}\int_{t-T_1}^{t+T_2} x_0(\tau-t_0)\,d\tau =$$

$$= \{\tau - t_0 = \tau'\} = \frac{1}{T_1+T_2}\int_{t-T_1+t_0}^{t+T_2-t_0} x_0(\tau')\,d\tau' = y_0(t-t_0) \Rightarrow \underline{\text{Invariante}}$$

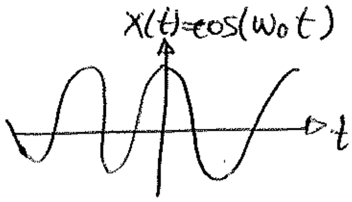
c) $h(t) = y(t)\Big|_{x(t)=\delta(t)} = \dfrac{1}{T_1+T_2}\underbrace{\displaystyle\int_{t-T_1}^{t+T_2} \delta(\tau)\, d\tau}_{} =$

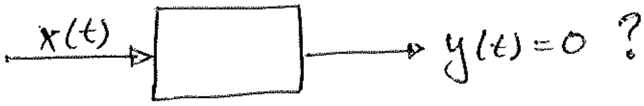
$$= \begin{cases} \dfrac{1}{T_1+T_2} & -T_2 < t < T_1 \\ 0 & \text{resto} \end{cases}$$

1 si $t - T_1 < 0$ y $t + T_2 > 0$
0 resto

$$= \frac{1}{T_1+T_2}\left[u(t+T_2) - u(t-T_1)\right]$$

d)





$y(t) = 0 \Leftrightarrow T_1 + T_2 = k\,\dfrac{2\pi}{w_0}$

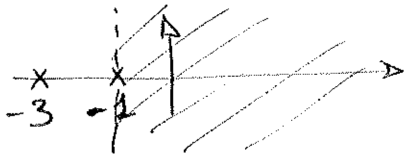
Feb 2002 P2

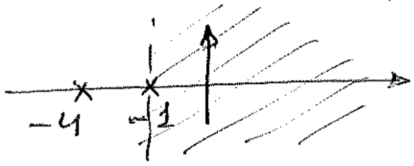
$x(t) = (e^{-t} + e^{-3t})\,u(t)$, $\quad y(t) = (2e^{-t} - 2e^{-4t})\,u(t)$

$y(t) = x(t) * h(t)$, ec. diferencial que rige el sistema??

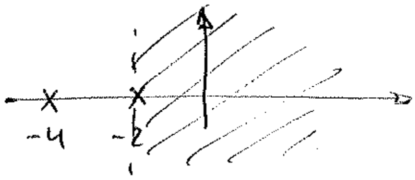
$Y(s) = X(s) \cdot H(s)$

$X(s) = \dfrac{1}{1+s} + \dfrac{1}{3+s} = \dfrac{4+2s}{(1+s)(3+s)}$; $Y(s) = \dfrac{2}{s+1} - \dfrac{2}{4+s} = \dfrac{6}{(1+s)(4+s)}$





$\dfrac{Y(s)}{X(s)} = \dfrac{6}{4+s} \cdot \dfrac{3+s}{4+2s} = \dfrac{9+3s}{(2+s)(4+s)}$



$H(s) = \dfrac{A}{2+s} + \dfrac{B}{4+s}$

$A = H(s)(2+s)\Big|_{s=-2}$

$B = H(s)(4+s)\Big|_{s=-4}$

$h(t) = A e^{-2t} u(t) + B e^{-4t} u(t)$

$$H(s) = \frac{9+3s}{s^2+6s+8}$$

$$Y(s)\cdot s^2 + Y(s)\cdot 6s + Y(s)\cdot 8 = 3s\cdot X(s) + 9\cdot X(s)$$

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 3\frac{dx(t)}{dt} + 9x(t) \quad , \text{ CRI}$$

b) $$h_T(t) = \sum_{k=0}^{\infty} a^k h(t-kT)$$

$$H_T(jw) = \sum_{k=0}^{\infty} a^k H(jw)e^{-jwkT} = H(jw) \underbrace{\sum_{k=0}^{\infty}\left(a\cdot e^{-jwT}\right)^k}_{\text{converge si y sólo si } |a\cdot e^{-jwT}|<1} =$$

$$= H(jw)\cdot\frac{1}{1-ae^{-jwT}}$$

$$|a|<1$$

c) $$H_i(jw) = \frac{(4+jw)(2+jw)}{3(3+jw)}\cdot\left(1-ae^{-jwT}\right)$$

No lo podremos calcular en $H(s)$ porque la descomposición en fracciones simples sólo aplica al cociente de polinomios. Separamos $H_i(s)$:

$$H_i(s) = \frac{(4+s)(2+s)}{3(3+s)}\cdot\left(1-ae^{-sT}\right) = H_1(s)\cdot H_2(s)$$

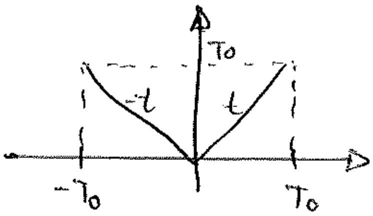
$$h_i(t) = h_1(t) * \left(\delta(t) - a\delta(t-T)\right) = h_1(t) - a h_1(t-T)$$

$$H_1(s) = \frac{1}{3}\cdot\frac{s^2+6s+8}{s+3} \longrightarrow h_1(t) \qquad \text{Grado(Num)} > \text{Grado(Den)}$$

$$H_1(s) = \frac{1}{3}\left(s+3-\frac{1}{s+3}\right) \rightarrow h_1(t) = \frac{1}{3}\left(\delta'(t) + 3\delta(t) - e^{-3t}u(t)\right)$$

sep 2003

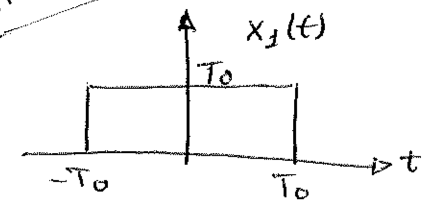
a) $x(t) = \begin{cases} |t| & |t| < T_0 \\ 0 & \text{resto} \end{cases}$



$$X(jw) = \int_{-\infty}^{+\infty} x(t)\, e^{-jwT}\, dt = -\int_{-T_0}^{0} t\, e^{-jwt}\, dt + \int_{0}^{T_0} t\, e^{-jwt}\, dt$$

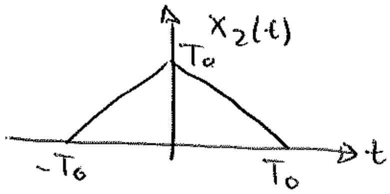
Habría que integrar por partes, pero saldría sin problemas.

Another way:



$$x(t) = x_1(t) - x_2(t)$$

$$X(jw) = T_0 \cdot \frac{2\,\text{sen}(wT_0)}{w} - \left(\frac{2\,\text{sen}(\frac{wT}{2})}{w}\right)^2$$



b) $y(t) = \sum_{K=-\infty}^{+\infty} x(t - T_0 - KT_1) = x(t-t_0) * \sum \delta(t - kT_1)$

$$Y(jw) = X(jw)\, e^{-jwt_0} \cdot \sum \frac{2\pi}{T_1}\, \delta\left(w - k\frac{2\pi}{T_1}\right) =$$

$$= \sum \frac{2\pi}{T_1}\, X\left(jk\frac{2\pi}{T_1}\right) e^{-jk\frac{2\pi}{T_1}T_0} \cdot \delta\left(w - k\frac{2\pi}{T_1}\right)$$

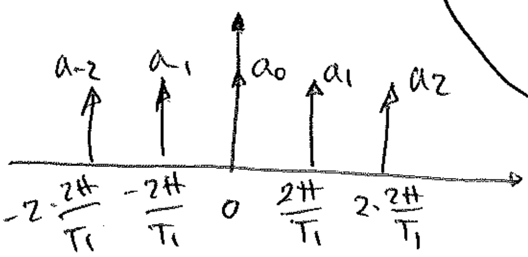
Another way:

$$Y(jw) = \sum 2\pi\, a_K\, \delta\left(w - k\frac{2\pi}{T_1}\right)$$

$$a_K = \left.\frac{X(jw)\, e^{-jwt_0}}{T_1}\right|_{w = \frac{k2\pi}{N_1}}$$

c) $\xrightarrow[Y(jw)]{y(t)} h(t) = \frac{d^2}{dt^2}\left(\frac{\text{sen}(Wt)}{\pi t}\right) \longrightarrow Z(jw) = \sum 2\pi\, b_K\, \delta\left(w - k\frac{2\pi}{T_1}\right)$

queremos eliminar $b_K \;\; \forall\, |K| > 2$



$$H(jw) = \begin{cases} -w^2 & |w| < W \\ 0 & \text{resto} \end{cases}$$

$$2 \cdot \frac{2\pi}{T_1} < W < 3 \cdot \frac{2\pi}{T_1}$$

$$b_k = a_k \cdot H(j\omega)\Big|_{\omega = k\frac{2\pi}{T_1}}$$

$$Z(j\omega) = X(j\omega) \cdot H(j\omega) = H(j\omega) \sum 2\pi a_k \, \delta\left(\omega - k\frac{2\pi}{T_1}\right) =$$

$$= \sum 2\pi a_k H(j\omega) \, \delta\left(\omega - k\frac{2\pi}{T_1}\right) =$$

$$= \sum 2\pi \, H\left(j\left(\omega - k\frac{2\pi}{T_1}\right)\right)$$

$$b_k = a_k \cdot \left(-\left(\frac{k2\pi}{T_1}\right)^2\right), \quad k \in (-2, -1, 1, 2)$$

Tema 3: TF en discreto

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n] = \sum_k x[n-k]\, h[k]$$

si $x[n] = e^{j\omega_0 n} \Rightarrow y[n] = \sum_k e^{j\omega_0 (n-k)} \cdot h[k] = e^{j\omega_0 n} \cdot \sum_k h[k] e^{-j\omega_0 k}$

$$H(e^{j\omega}) = \sum_k h[k] \cdot e^{-j\omega k}$$

si $x[n] = \sum_k \underbrace{a_k e^{j\omega_k n}} \Rightarrow y[n] = \sum a_k H(e^{j\omega_k}) e^{j\omega_k \cdot n}$

- si es aperiódico: TF
- si es periódico: DSF

$\rightarrow H(e^{j(\omega + 2\pi)}) = \sum_k h[k] e^{-j\omega k} \cdot \underbrace{e^{-j2\pi k}}_{1} = H(e^{j\omega})$

$H(e^{j\omega})$ es periódica de periodo 2π

luego toda transformada será periódica de periodo 2π

Ecuación de análisis:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

periódica 2π !!

Ecuación de síntesis:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Condiciones suficientes de existencia de la TF:

- $\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$
- continuidad en el dominio de la frecuencia
- acotación en el dominio de la frecuencia

Ej: $X(e^{jw}) = \sum_n x[n]\, e^{-jwn}$

$\rightarrow x[n] = \delta[n] \longrightarrow X(e^{jw}) = \sum_n \delta[n] \cdot e^{-jwn} = \sum_n \delta[n] \cdot \underbrace{\cancel{e^{-jw0}}}_{1} = 1$

$\rightarrow x[n] = a^n u[n] \;;\; |a| < 1$

$$X(e^{jw}) = \sum_n a^n u[n]\, e^{-jwn} = \sum_{n=0}^{+\infty} a^n e^{-jwn} = \sum_{n=0}^{+\infty} (a e^{-jw})^n =$$

$$= \left\{ \begin{matrix} \text{serie} \\ \text{geom.} \end{matrix} \right\} = \frac{1}{1 - a e^{-jw}}$$

$\rightarrow x[n] = a^{|n|} \;,\; |a| < 1 \;;\; x[n] = a^n + a^{-n}$

$$X(e^{jw}) = \sum_n a^{|n|} e^{-jwn} = \sum_{n=-\infty}^{-1} a^{-n} e^{-jwn} + \sum_{n=0}^{+\infty} a^n e^{-jwn} =$$

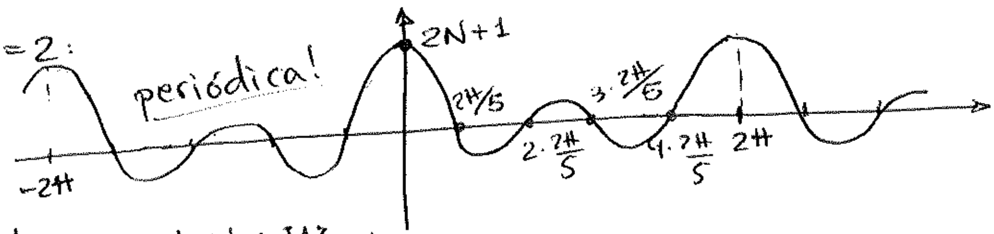
$$= \sum_{n=1}^{\infty} (a e^{jw})^n + \sum_{0}^{\infty} (a e^{-jw})^n = \frac{a e^{jw}}{1 - a e^{jw}} + \frac{1}{1 - a e^{-jw}}$$

$\rightarrow x[n] = \begin{cases} 1 & , -N \leq n \leq N \\ 0 & , \text{resto} \end{cases}$

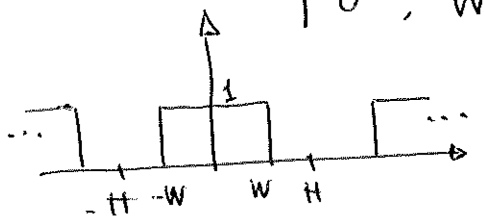
$$X(e^{jw}) = \sum_{n=-N}^{+N} 1 \cdot e^{-jwn} = \frac{e^{+jwN} - e^{-jw(N+1)}}{1 - e^{-jw}} =$$

$$= \frac{e^{jwN} \cdot e^{jw/2}\, \cancel{e^{-jw/2}} - e^{-jwN}\, e^{-jw/2}\, \cancel{e^{-jw/2}}}{e^{jw/2} \cdot \cancel{e^{-jw/2}} - e^{-jw/2} \cdot \cancel{e^{-jw/2}}} \cdot \frac{2j}{2j} = \frac{\text{sen}(w(N+\frac{1}{2}))}{\text{sen}(w/2)}$$

sea $N = 2$:

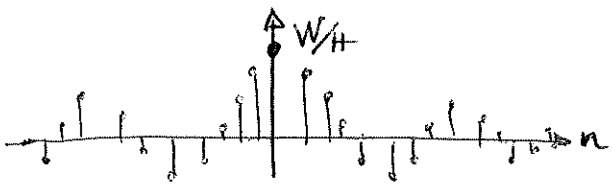


$\rightarrow X(e^{jw}) = \begin{cases} 1 & , 0 < |w| < W < \pi \\ 0 & , W < |w| < \pi \end{cases}$ periódica 2π

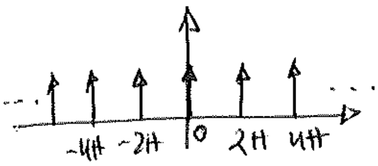


$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw = \frac{1}{2\pi} \int_{-W}^{W} e^{jwn}\, dw$$

$$= \frac{1}{2\pi j n} \left(e^{jWn} - e^{-jWn} \right) = \frac{\text{sen}(Wn)}{\pi n}$$



$$\rightarrow X(e^{jw}) = \sum 2\pi\,\delta(w - k2\pi)$$



$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})\, e^{jwn}\, dw =$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\,\delta(w)\,\underbrace{\cancel{e^{jwn}}}_{1}\, dw = 1$$

$$\rightarrow X(e^{jw}) = TF(\cos(w_0 n)) = \frac{1}{2}\,TF(e^{jw_0 n}) + \frac{1}{2}\,TF(e^{-jw_0 n}) =$$

$$= \frac{2\pi}{2}\left(\sum_{k=-\infty}^{+\infty} \delta(w - w_0 - k2\pi) + \delta(w + w_0 - k2\pi)\right)$$

$$\rightarrow X(e^{jw}) = \frac{1}{2j}\,TF(e^{jw_0 n}) - \frac{1}{2j}\,TF(e^{-jw_0 n}) = TF(\text{sen}(w_0 n)) =$$

$$= \frac{2\pi}{2j}\sum_{k}\left(\delta(w - w_0 - k2\pi) - \delta(w + w_0 - k2\pi)\right)$$

Desarrollo en Serie de Fourier:

$x[n] = x[n+N] \Rightarrow x$ es periódica ; $w = \frac{2\pi}{N}$

$$x[n] = \sum_{n=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n} \quad ; \quad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

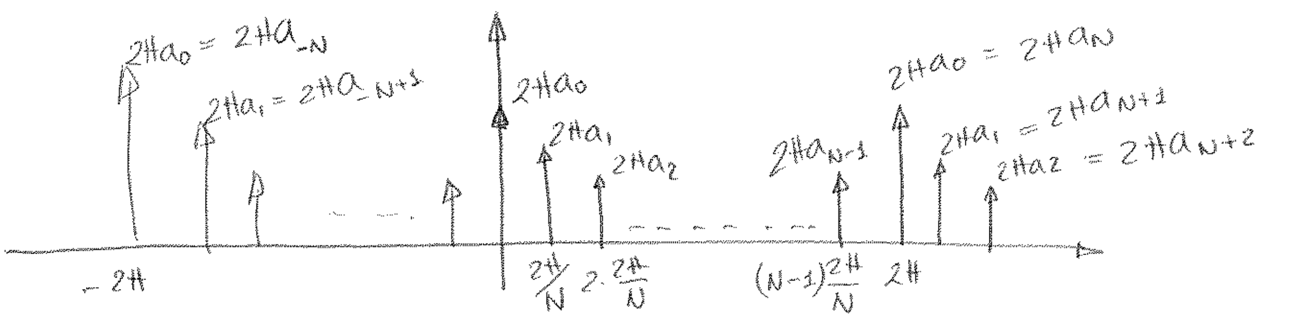
$$x[n] = \sum_{k=-\infty}^{+\infty} x_p[n - kN] \qquad a_{k+N} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}\, e^{-jN\frac{2\pi}{N}n} = a_k$$

periódicos de periodo N

$$X(e^{jw}) = TF(x[n]) = TF\left(\sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}\right) =$$

$$= \sum_{k=\langle N\rangle} a_k\, TF(e^{jk\frac{2\pi}{N}n}) = \sum_{k=\langle N\rangle} a_k \sum_{m=-\infty}^{+\infty} 2\pi\,\delta\left(w - \frac{k2\pi}{N} + m2\pi\right)$$

$$e^{jw_0 n} \longleftrightarrow \sum_{m=-\infty}^{+\infty} 2\pi\,\delta(w - w_0 + m2\pi)$$



Análisis de SLI en Fourier

$$x[n] \leftrightarrow X(e^{jw}) \longrightarrow \boxed{h[n] \longleftrightarrow H(e^{jw})} \longrightarrow y[n] \longleftrightarrow Y(e^{jw})$$

$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

señales periódicas:

$$z[n] = \sum_{k=-\infty}^{+\infty} x[n-kN] \longrightarrow \begin{array}{l} w[n] = x[n] * h[n] \\ y[n] = \sum_k w[n-kN] \end{array}$$

$$z[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n} \longrightarrow y[n] = \sum_{k=\langle N\rangle} a_k \cdot H(e^{jw})\Big|_{w=k\frac{2\pi}{N}} e^{jk\frac{2\pi}{N}n}$$

$$a_k = \frac{X(e^{jw})}{N}\Bigg|_{w=k\frac{2\pi}{N}} \qquad = \sum_{k=\langle N\rangle} b_k\, e^{jk\frac{2\pi}{N}n}$$

$$Y(e^{jw}) = Z(e^{jw})\,H(e^{jw}) = H(e^{jw}) \cdot \sum_k 2\pi a_k\, \delta\left(w - k\frac{2\pi}{N}\right)$$

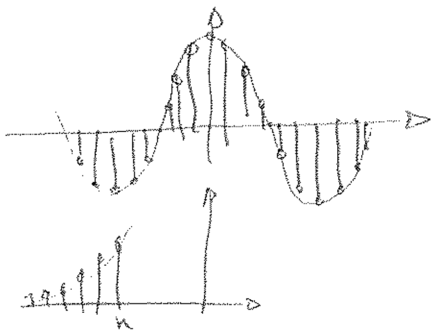
$$= \sum_k 2\pi a_k\, H\left(e^{jk\frac{2\pi}{N}}\right) \delta\left(w - \frac{k2\pi}{N}\right)$$

Ej: $x[n] = \cos\left(\frac{\pi}{3}n\right)$

$h[n] = \left(\frac{1}{2}\right)^n u[n]$

$y[n] = x[n] * h[n]$

$H(e^{jw}) = \dfrac{1}{1 - \frac{1}{2}e^{-jw}}$



por ser periódica tiene 1 sólo intervalo

$$y[n] = \frac{1}{2} H(e^{j\pi/3})\, e^{j\pi/3\, n} + \frac{1}{2} H(e^{-j\pi/3})\, e^{-j\pi/3\, n} =$$

$$= \frac{1}{2} A e^{j(\pi/3\, n + \theta)} + \frac{1}{2} A e^{-j\left(\frac{\pi}{3}n + \theta\right)} = A\cos\left(\frac{\pi}{3}n + \theta\right)$$

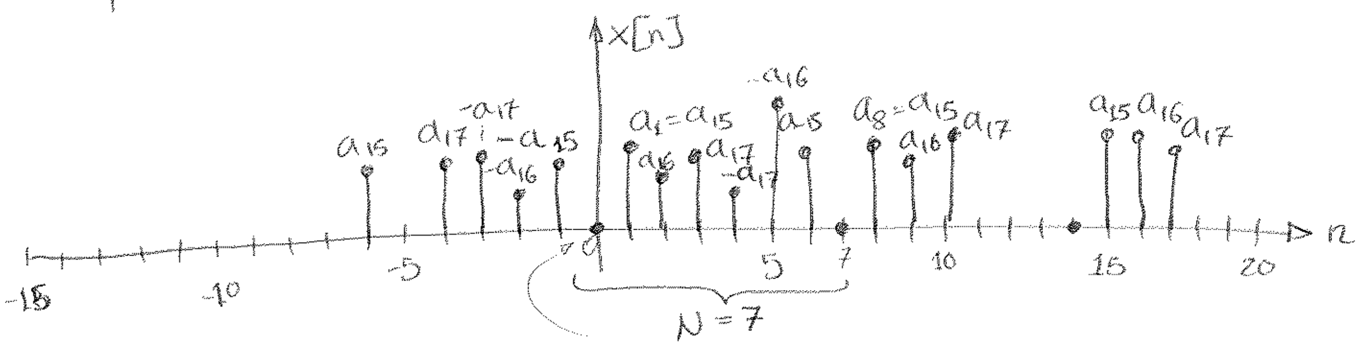
$A e^{j\theta}$

Ej 3.10

$x[n]$ real, impar, periódica $N=7$; conocemos a_{15}, a_{16}, a_{17} — calcule $a_0, a_{-1}, a_{-2}, a_{-3}$

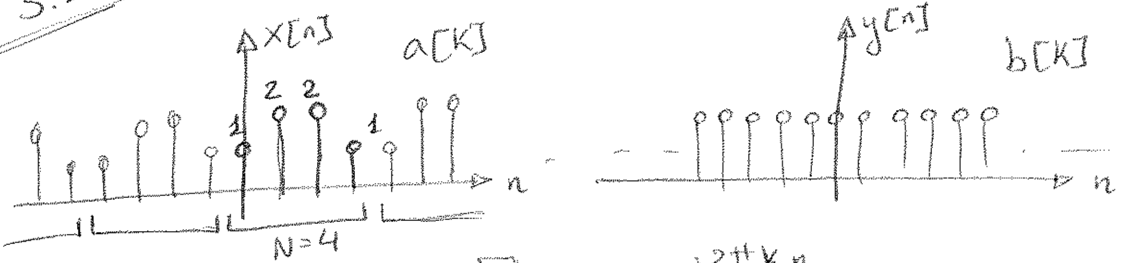
Real, periód. $\Rightarrow a_k = a_{k+N}$

impar $\Rightarrow a_k = -a_{-k}$



$a_k = -a_{-k} \Leftrightarrow a_0 = -a_0 = 0$

Ej 3.12



$z[n] = x[n]\cdot y[n] = \sum_{k=\langle N\rangle} c_k\, e^{j\frac{2\pi}{N}kn}$

$c_k = a_k * b_k \xrightarrow[\text{periódicos}]{\text{por ser}} c_k = a_k \circledast b_k$

$c_k = \sum_{m=\langle N\rangle} a[m]\cdot b[k-m] = \sum_{m=-\infty}^{+\infty} a_p[m]\, b[k-m]$; a_p = periodo de a_k

$= a_p[k] * b[k] = 6 \quad \forall k$

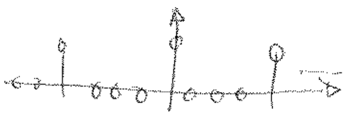
Ej 3.48:

$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}$

$y[n] = g(x[n]) = \begin{cases} x[n-n_0] = \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}(n-n_0)} = \sum a_k e^{-jk\frac{2\pi}{N}n_0}\cdot e^{jk\frac{2\pi}{N}n} \\ x[n]-x[n-1] = \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n} - \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}(n-1)} \\ x^*[-n] = \left(\sum_{k=\langle N\rangle} a_k e^{-jk\frac{2\pi}{N}n}\right)^* = \sum_{k=\langle N\rangle} a_k^*\, e^{jk\frac{2\pi}{N}n} \end{cases}$

Ej 3.14:

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-4k] \longrightarrow \boxed{H(e^{jw})?} \longrightarrow y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right)$$



$$X(e^{jw}) = \sum 2\pi a_k \, \delta\left(w - k\frac{2\pi}{N}\right) \; ; \; a_k = \frac{1}{N} = \frac{1}{4}$$

$$Y(e^{jw}) = \sum 2\pi b_k \, \delta\left(w - k\frac{2\pi}{N}\right)$$

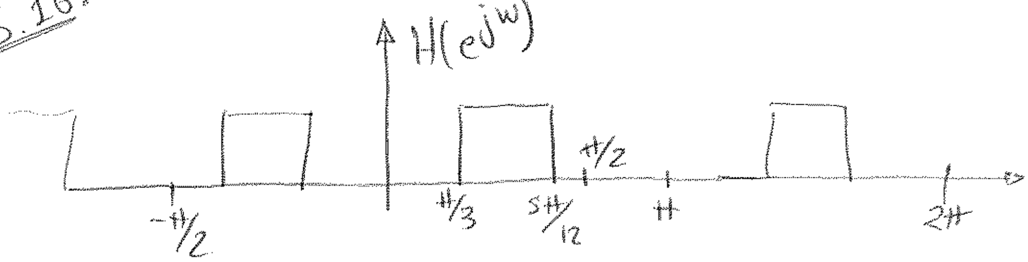
$$y[n] = \underbrace{\frac{e^{j\pi/4}}{2}}_{b_5 = b_1} \cdot e^{j5\frac{2\pi}{4}n} + \underbrace{\frac{e^{-j\pi/4}}{2}}_{b_{-5} = b_{-1} = b_3} e^{-j5\frac{2\pi}{4}n} = \sum_{k=\langle N \rangle} b_k e^{jk\frac{2\pi}{N}n}$$

$$\begin{cases} b_0 = 0 \\ b_1 = \dfrac{e^{j\pi/4}}{2} \\ b_2 = 0 \\ b_3 = \dfrac{e^{-j\pi/4}}{2} \end{cases}$$

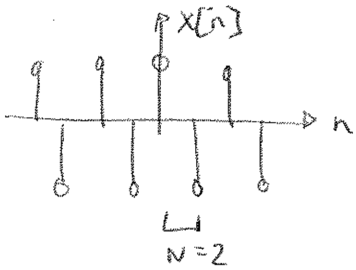
$$Y(e^{jw}) = H(e^{jw}) \cdot X(e^{jw}) = \sum 2\pi a_k H(e^{jw}) \cdot \delta\left(w - k\frac{2\pi}{N}\right) = \sum 2\pi a_k \, H\left(e^{jk\frac{2\pi}{N}}\right)\delta\left(w - k\frac{2\pi}{N}\right)$$

$$b_k = a_k \, H\left(e^{jk\frac{2\pi}{N}}\right)$$

Ej 3.16:

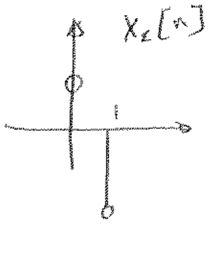


$$x[n] = (-1)^n \longrightarrow \boxed{H(e^{jw})} \longrightarrow y[n]$$



$$X(e^{jw}) = \sum 2\pi a_k \, \delta\left(w - k\frac{2\pi}{2}\right)$$

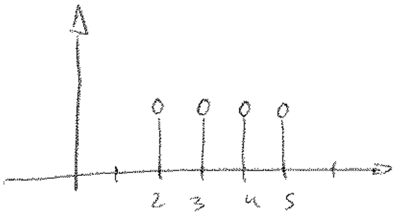
$\downarrow$



$$\longrightarrow X_1(e^{jw}) = 1 - e^{-jw} \Rightarrow a_k = \left.\frac{X_1(e^{jw})}{N}\right|_{w = \frac{k2\pi}{N}}$$

Ej 5.21:

a) $x[n] = u[n-2] - u[n-6]$; $X(e^{jw}) = \sum_n x[n] e^{-jwn}$



$$X(e^{jw}) = \sum_{2}^{5} e^{-jwn} = \frac{e^{-j2w} - e^{-jw6}}{1 - e^{-jw}} =$$

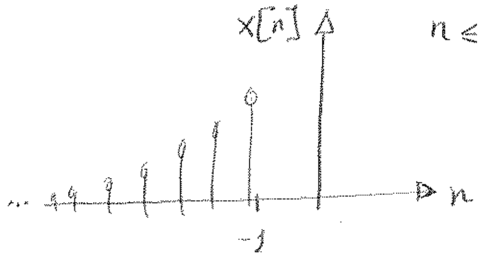
$$= \frac{e^{-j4w}\left(e^{j2w} - e^{-j2w}\right)}{e^{jw/2}e^{-jw/2} - e^{-jw/2}e^{-jw/2}} =$$

$$= \frac{e^{-j4w}}{e^{-jw/2}} \cdot \frac{2j}{2j} \cdot \frac{e^{j2w} - e^{-j2w}}{e^{jw/2} - e^{-jw/2}} = e^{-j\frac{w7}{2}} \cdot \frac{\operatorname{sen}(2w)}{\operatorname{sen}(w/2)}$$

b) $x[n] = \left(\frac{1}{2}\right)^{-n} \cdot u[\underbrace{-n-1}]$ $\longrightarrow X(e^{jw}) = \sum_n x[n] e^{-jwn}$

$-n-1 \geq 0$

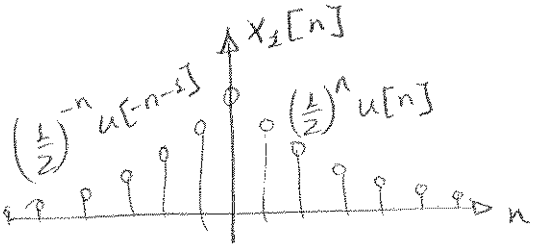
$n \leq -1$



$$X(e^{jw}) = \sum_{-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{-jwn} = \sum_{1}^{\infty}\left(\frac{1}{2}e^{jw}\right)^{n} =$$

$$= \frac{\frac{1}{2}e^{jw}}{1 - \frac{1}{2}e^{jw}}$$

c) $x[n] = \underbrace{\left(\frac{1}{2}\right)^{|n|}}_{x_1[n]} \underbrace{\cos\left(\frac{\pi}{8}n - \frac{\pi}{8}\right)}_{x_2[n]}$



$$X_1(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}} + \frac{1/2\, e^{jw}}{1 - \frac{1}{2}e^{jw}}$$

$$x[n] = x_1[n] \cdot \left(\frac{e^{j\left(\frac{\pi}{8}n - \frac{\pi}{8}\right)} + e^{-j\left(\frac{\pi}{8}n - \frac{\pi}{8}\right)}}{2}\right)$$

$$= \frac{e^{-j\pi/8}}{2} x_1[n] e^{j\frac{\pi}{8}n} + \frac{e^{j\pi/8}}{2} x_1[n] e^{-j\frac{\pi}{8}n}$$

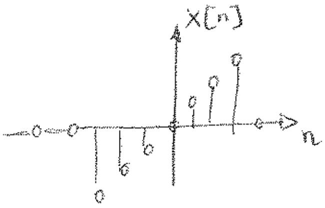
$$\hookrightarrow X(e^{jw}) = \frac{e^{-j\pi/8}}{2} X_1\left(e^{j(w-\pi/8)}\right) + \frac{e^{j\pi/8}}{2} X_1\left(e^{j(w+\pi/8)}\right)$$

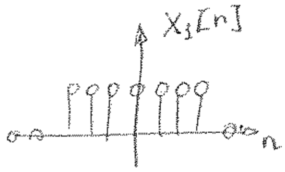
$$\hookrightarrow X(e^{jw}) = \frac{1}{\cancel{2\pi}}\left(\frac{e^{-j\pi/8}}{2} X_1(e^{jw}) \circledast \sum_{k=-\infty}^{+\infty} \cancel{2\pi}\delta\left(w - \frac{\pi}{8} + 2k\pi\right) + \frac{e^{j\pi/8}}{2} X_1(e^{jw}) \circledast \sum_k \cancel{2\pi}\delta(\ldots\right.$$

$$= \frac{e^{-j\pi/8}}{2} X_1(e^{jw}) * \delta\left(w - \frac{\pi}{8}\right) + \frac{e^{j\pi/8}}{2} X_1(e^{jw}) * \delta\left(w + \frac{\pi}{8}\right) =$$

$$= \frac{e^{-j\pi/8}}{2} X_1\left(e^{j(w-\pi/8)}\right) + \frac{e^{j\pi/8}}{2} X_1\left(e^{j(w+\pi/8)}\right)$$

j) $x[n] = \begin{cases} n & -3 \le n \le 3 \\ 0 & \text{resto} \end{cases}$



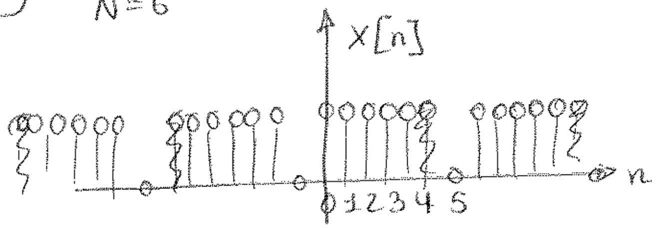


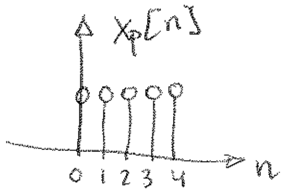
$x_1[n] \longleftrightarrow X_1(e^{jw}) = \frac{sen(w(N_1+\frac{1}{2}))}{sen(w/2)} = \frac{sen(7w/2)}{sen(w/2)}$

$x[n] = n \cdot x_1[n]$

$$X(e^{jw}) = j\,\frac{dX(e^{jw})}{dw} = j\,\frac{\frac{7}{2}\cos(\frac{7w}{2})\,sen(w/2) - \frac{1}{2}\cos(w/2)\,sen(7w/2)}{sen^2(w/2)}$$

i) N=6





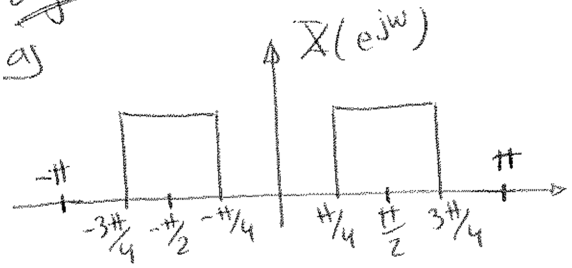
$$X(e^{jw}) = \sum 2\pi a_k\, \delta\left(w - k\frac{2\pi}{N}\right)$$

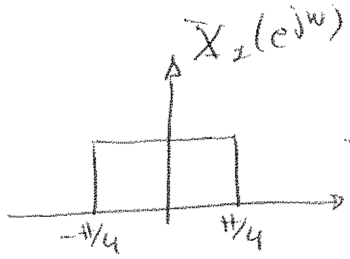
$$a_k = \frac{X_p(e^{jw})}{N}\Big|_{w = \frac{k2\pi}{N}}$$

$$X_p(e^{jw}) = \frac{sen(\frac{5w}{2})}{sen(\frac{w}{2})}\, e^{-j2w}$$

Ej 5.22:

a)





$\rightarrow x_1[n] = \frac{sen(\frac{\pi}{4}n)}{\pi n}$

$$X(e^{jw}) = X_1(e^{j(w-\pi/2)}) + X_1(e^{j(w+\pi/2)})$$

$$x[n] = x_1[n]\,e^{j\frac{\pi}{2}n} + x_1[n]\,e^{-j\frac{\pi}{2}n} = 2\cos\left(\frac{\pi}{2}\right) \cdot \frac{sen(\pi/4\; n)}{\pi n}$$

Ej 5.22:

b) $X(e^{jw}) = 1 + 3e^{-jw} + 2e^{-j2w} - 4e^{-j3w} + e^{-j10w}$

$$x[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2] - 4\delta[n-3] + \delta[n-10]$$

c) $X(e^{jw}) = e^{-jw/2}, \quad -\pi < w < \pi$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\frac{w}{2}} e^{jwn}\, dw = \frac{1}{2\pi j(n-1/2)}\left(e^{j(n-1/2)\pi} - e^{-j(n-1/2)\pi}\right)$$

$$= \frac{\operatorname{sen}((n-1/2)\pi)}{\pi(n-1/2)}$$

h) $X(e^{jw}) = \dfrac{1-\left(\frac{1}{3}\right)^6 e^{-jw6}}{1-\frac{1}{3}e^{-jw}}$

$$X(e^{jw}) = \frac{1}{1-\frac{1}{3}e^{-jw}} - \left(\frac{1}{3}\right)^6 \frac{1}{1-\frac{1}{3}e^{-jw}} e^{-jw6}$$

$$x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^6\left(\frac{1}{3}\right)^{n-6} u[n-6] = \left(\frac{1}{3}\right)^n (u[n]-u[n-6])$$

Transformada Z (bilateral)

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}, \quad r_1 < |z| < r_2 \; ; \quad x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz$$

ec. de análisis — ec. de síntesis

Además de la expresión necesitamos hallar la ROC:

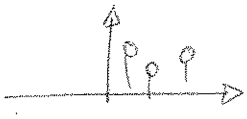
- las fronteras son circunferencias centradas en $z=0$
- si $x[n]$ es indefinida a derechas $\Rightarrow$ ROC = exterior a una circunf
- si $x[n]$ es indefinida a izquierdas $\Rightarrow$ ROC = interior a una circunf
- si $x[n]$ es indef. ambos lados $\Rightarrow$ ROC = corona circular
- si $x[n]$ es finita y absolutamente sumable $\Rightarrow$ ROC = todo plano*

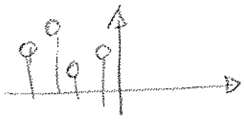
$$TF(x[n]) = Z(x[n])\Big|_{|z|=1}$$

circunferencia radio 1

$z = e^{jw}$

* si $x[n]$ es de duración finita $\sum |x[n]| < \infty$
entonces ROC es todo el plano excepto 0 ó ∞

si  ROC $\Rightarrow$ todo el plano $- \{0\}$

si  ROC $\Rightarrow$ todo el plano $- \{\infty\}$

si  ROC $\Rightarrow$ todo el plano $- \{0, \infty\}$

Algunas propiedades:

linealidad: $x[n] = x_1[n] + x_2[n] \longleftrightarrow X(z) = X_1(z) + X_2(z)$
$$R \supset R_1 \cap R_2$$

desplaz. temp: $x[n] = x_1[n-n_0] \longleftrightarrow X(z) = z^{-n_0} X_1(z) \quad R \subset R_1$

convolución: $x[n] = x_1[n] * x_2[n] \longleftrightarrow X(z) = X_1(z) \cdot X_2(z)$
$$R \supset R_1 \cap R_2$$

dif. temporal: $x_1[n] - x_1[n-1] \longleftrightarrow (1 - z^{-1}) X_1(z), \quad R \subset R_1$

derivada: $n\, x_1[n] \longleftrightarrow -z \dfrac{dX_1(z)}{dz}, \quad R = R_1$

Algunas transformadas:

→ $\delta[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1$, ROC = todo el plano

→ $\delta[n-n_0] \longleftrightarrow \sum_{n=-\infty}^{+\infty} \delta[n-n_0] z^{-n} = \sum \delta[n-n_0] \cdot z^{-n_0} = z^{-n_0} \quad \forall z \neq 0$

$\delta[n+n_0] \longleftrightarrow \sum_{-\infty}^{+\infty} \delta[n+n_0] z^{n} = \sum_{-\infty}^{+\infty} \delta[n+n_0] z^{n_0} = z^{n_0} \quad \forall z \neq \infty$

→ $a^n u[n] \longleftrightarrow \sum_{-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{0}^{\infty} (a z^{-1})^n = \dfrac{1}{1 - a z^{-1}} \quad |z| > |a|$

→ $-a^n u[-n-1] \longleftrightarrow -\sum_{-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{-\infty}^{-1} a^n z^{-n} = \sum_{1}^{\infty} (a^{-1} z)^n = \dfrac{1}{1 - a z^{-1}}$
$$|z| < |a|$$

→ $(n+1) a^n u[n] \longleftrightarrow \dfrac{1}{(1 - a z^{-1})^2} \quad |z| > |a|$

→ $-(n+1) a^n u[-n-1] \longleftrightarrow \dfrac{1}{(1 - a z^{-1})^2} \quad |z| < |a|$

transformación inversa:

descomposición en fracciones simples

→ polos simples:

$$\frac{1}{1-az^{-1}}$$

$|z|<|a| \;\to\; -a^n u[-n-1]$

$|z|>|a| \;\to\; a^n u[n]$

→ polos dobles:

$$\frac{1}{(1-az^{-1})^2}$$

$|z|<|a| \;\to\; -(n+1)a^n u[-n-1]$

$|z|>|a| \;\to\; (n+1)a^n u[n]$

Análisis de sistemas descritos por ecuaciones en diferencias

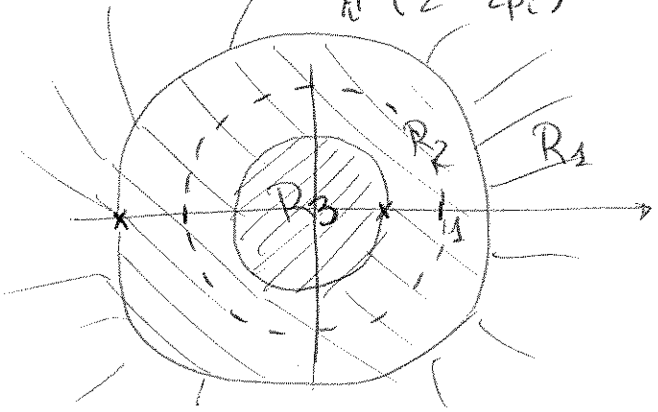
$$x[n] \to \boxed{\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k] \quad H(e^{jw}) \quad H(z)} \longrightarrow y[n]$$

$$Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

$$\sum_{k=0}^{N} a_k\, Y(z)\, z^{-k} = \sum_{k=0}^{M} b_k\, X(z)\, z^{-k} \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = A\cdot\frac{\prod_i (z-z_{ci})}{\prod_i (z-z_{pi})} \Rightarrow \begin{cases} z_{ci} = \text{ceros} \\ z_{pi} = \text{polos} \end{cases}$$

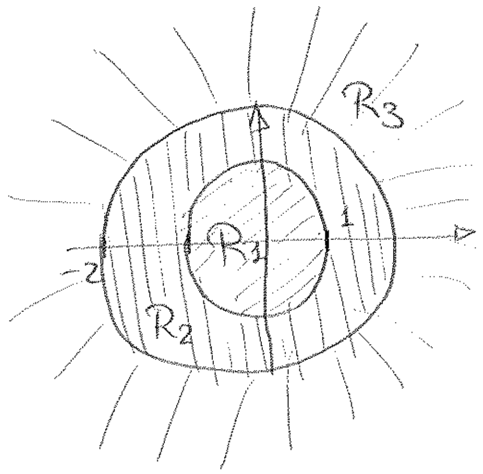
los polos marcan las circunferencias que dividen las ROCs



existirá $H(e^{jw}) = H(z)\big|_{z=e^{jw}}$ sólo en la ROC ⊂ circunf radio 1 (en el ejemplo sólo en R_2)

Ej 10.9:

$$X(z) = \frac{1 - \frac{1}{3} z^{-1}}{(1-z^{-1})(1+2z^{-1})}$$



polo $\Rightarrow z^{-1} = 1 \Rightarrow z = 1$

polo $\Rightarrow 2z^{-1} = -1 \Rightarrow z = -2$

polos simples:

$$\frac{1}{1-az^{-1}}$$

$|z|<|a|$: $-a^n u[-n-1]$

$|z|>|a|$: $a^n u[n]$

$$X(z) = \frac{1 - \frac{1}{3} z^{-1}}{(1-1\cdot z^{-1})(1-(-2)z^{-1})}$$

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1+2z^{-1}}$$

$\frac{A}{1-z^{-1}}$: $|z|<1$: $-A\,u[n-1]$; $|z|>1$: $A\,u[n]$

$\frac{B}{1+2z^{-1}}$: $|z|<2$: $-B(-2)^n u[-n-1]$; $|z|>2$: $B(-2)^n u[n]$

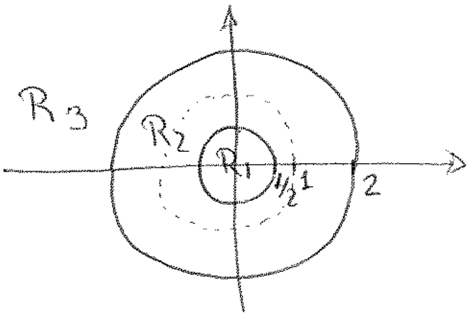
$$h_1[n] = -A\,u[-n-1] - B(-2)^n u[-n-1]$$
$$h_2[n] = A\,u[n] - B(-2)^n u[-n-1]$$
$$h_3[n] = A\,u[n] + B(-2)^n u[n]$$

Ej 10.35:

$$x[n] \longrightarrow \boxed{y[n-1] - \frac{5}{2} y[n] + y[n+1] = x[n]} \longrightarrow y[n]$$

$$Y(z)\,z^{-1} - \frac{5}{2} Y(z) + Y(z)\,z = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{5}{2} + z} = \frac{z^{-1}}{z^{-2} - \frac{5}{2} z^{-1} + 1} \longrightarrow z^{-1} \begin{cases} 2 \Rightarrow z = \frac{1}{2} \\ \frac{1}{2} \Rightarrow z = 2 \end{cases}$$



$$H(z) = \frac{z^{-1}}{(z^{-1}-2)(z^{-1}-\frac{1}{2})} = \frac{z^{-1}\cdot(-\frac{1}{2})(-2)}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-2z^{-1}}$$

$$H(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-2z^{-1}}$$

$|z|<1/2$: $-A\left(\frac{1}{2}\right)^n u[-n-1]$ $\quad$ $|z|>1/2$: $A\left(\frac{1}{2}\right)^n u[n]$ $\quad$ $|z|<2$: $-B\,2^n u[-n-1]$ $\quad$ $|z|>2$: $B\,2^n u[n]$

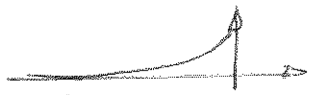
$h_1[n] = -A\left(\frac{1}{2}\right)^n u[-n-1] - B\,2^n u[-n-1]$ anticausal, no estable

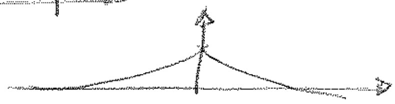
$h_2[n] = A\left(\frac{1}{2}\right)^n u[n] - B\,2^n u[-n-1]$ no causal, estable

$h_3[n] = A\left(\frac{1}{2}\right)^n u[n] + B\,2^n u[n]$ causal, no estable

- Causalidad:

causal: $u[n]$ 

anticausal: $u[-n-1]$ 

no causal $u[n]$, $u[-n-1]$ 

- Estabilidad:

$\sum |h[n]| < \infty$

$\sum |h[n]|\, r^{-n} < \infty$

sólo en la región estable:

$$H(e^{jw}) = H(z)\big|_{z=e^{jw}}$$

en las demás regiones $\nexists H(e^{jw})$

$$A \Rightarrow H(z)\left(1-\frac{1}{2}z^{-1}\right) = \frac{z^{-1}\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} = A + B\frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-2z^{-1}\right)}\Bigg|_{z^{-1}=2}$$

$$B \Rightarrow H(z)\left(1-2z^{-1}\right) = \frac{z^{-1}\left(1-2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} = A\frac{\left(1-2z^{-1}\right)}{1-\frac{1}{2}z^{-1}} + B\Bigg|_{z^{-1}=\frac{1}{2}}$$

Transformada Z unilateral

Ecuaciones dif. con condiciones iniciales NO nulas

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

si se trata de secuencias causales: $x[n] = 0 \,,\; n<0$
es lo mismo que la bilateral

si se trata de secuencias no causales:

$$y[n] = x[n-n_0] \longleftrightarrow Y(z) = z^{-n_0} X(z) + \sum_{l=1}^{n_0} z^{l-n_0}\, x[-l]$$

En el caso de ec. diferenciales:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$\hookrightarrow$ CRI: $\displaystyle\sum_{k=0}^{N} a_k\, Y(z)\, z^{-k} = \sum_{k=0}^{M} b_k\, X(z)\, z^{-k}$

$\hookrightarrow$ No RI: $\displaystyle\sum_{k=0}^{N} a_k\left(Y(z)\, z^{-k} + \sum_{l=1}^{k} y(-l)\, z^{l-k}\right) = \sum_{k=0}^{M} b_k\left(X(z)\, z^{-k} + \sum_{l=1}^{k} x(-l) \cdot z^{l-k}\right)$

Ej: $y[n] - \frac{1}{2} y[n-2] = x[n]$

$\hookrightarrow$ CRI $\Rightarrow$ $Y(z) - \frac{1}{2} Y(z)\, z^{-2} = X(z)$

$$Y(z) = X(z) \cdot \frac{1}{1-\frac{1}{2}z^{-2}}$$

$\hookrightarrow$ No RI $\Rightarrow$ $Y(z) - \frac{1}{2}\left(Y(z)\, z^{-2} + y(-1)\, z^{-1} + y(-2)\right) = X(z)$

$$Y(z)\left(1-\frac{1}{2}z^{-2}\right) = X(z) + \frac{1}{2}\left(\alpha z^{-1} + \beta\right)$$

$$Y(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-2}}\, X(z)}_{Y(z)\ \text{CRI}} + \underbrace{\frac{1}{2}\, \frac{\alpha z^{-1}+\beta}{1-\frac{1}{2}z^{-2}}}_{Y(z)\ \text{NO CRI}}$$

$$y[n] = y_{CRI}[n] + y_{NO\,CRI}[n]$$

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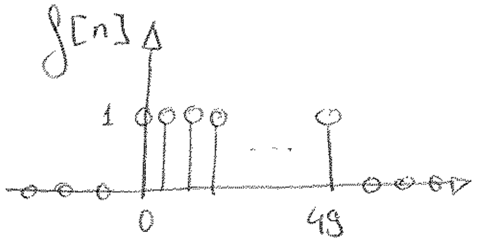


a)

$$F(e^{jw}) = \sum_{n=-\infty}^{+\infty} f[n]\, e^{-jwn} = \sum_{0}^{49} e^{-jwn} = \frac{1 - e^{-jw50}}{1 - e^{-jw}} =$$

$$= \frac{e^{jw25} e^{-jw25} - e^{-jw25} e^{-jw25}}{e^{jw/2} e^{-jw/2} - e^{-jw/2} e^{-jw/2}} = \frac{e^{-jw25}}{e^{-jw/2}} \cdot \frac{e^{jw25} - e^{-jw25}}{e^{jw/2} - e^{-jw/2}} \cdot \frac{2j}{2j} =$$

$$= e^{-j\frac{49}{2}w} \cdot \frac{\operatorname{sen}(w25)}{\operatorname{sen}(w/2)}$$

a2)

$$x[n] = \sum_{K=-\infty}^{+\infty} f[n-100K] \Rightarrow X(e^{jw}) = \sum 2\pi a_K \delta(w - k\tfrac{2\pi}{100})$$

Poisson: $a_K = \left.\frac{F(e^{jw})}{100}\right|_{w = K\frac{2\pi}{100}}$

b)

$$y[n] - \tfrac{1}{2} y[n-2] = x[n] - \tfrac{1}{4} x[n-4] \qquad \text{LTI causal CRI}$$

$$Y(z) - \tfrac{1}{2} Y(z) z^{-2} = X(z) - \tfrac{1}{4} X(z) z^{-4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 1/4\, z^{-4}}{1 - \frac{1}{2} z^{-2}} = \frac{\cancel{(1 - \frac{1}{2} z^{-2})}(1 + \frac{1}{2} z^{-2})}{\cancel{(1 - \frac{1}{2} z^{-2})}} = 1 + \tfrac{1}{2} z^{-2}$$

$$h[n] = \delta[n] + \tfrac{1}{2}\delta[n-2] \rightarrow \text{ROC} = \text{todo plano} - \{0\}$$

c)

$$x[n] \longrightarrow h[n] \longrightarrow y[n] = x[n] + \tfrac{1}{2} x[n-2]$$

$$y[n] = \sum_{K=\langle N\rangle} b_K\, e^{jk\frac{2\pi}{N}n}$$

$$\hookrightarrow b_K = a_K H(e^{jw})\Big|_{w = k\frac{2\pi}{N}}$$

feb02

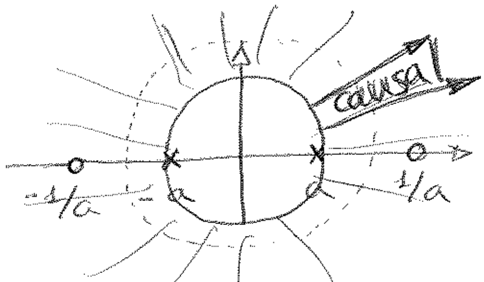
LTI causal CRI $a \in (0,1)$

$$\xrightarrow{x[n]} y[n] - a^2 y[n-2] = x[n-2] - a^2 x[n] \xrightarrow{y[n]}$$

$$Y(z) - a^2 Y(z) z^{-2} = X(z) z^{-2} - a^2 X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2} - a^2}{1 - a^2 z^{-2}} = \frac{(z^{-1} - a)(z^{-1} + a)}{(1 - a z^{-1})(1 + a z^{-1})}$$

$$H(e^{jw}) = \frac{e^{-j2w} - a^2}{1 - a^2 e^{-jw2}}$$



$$|H(e^{jw})|^2 = H(e^{jw}) \cdot H^*(e^{jw}) = \frac{(e^{-j2w} - a^2)}{(1 - a^2 e^{-j2w})} \cdot \frac{(e^{j2w} - a^2)}{(1 - a^2 e^{j2w})} =$$

$$= \frac{1 + a^4 - a^2(e^{j2w} + e^{-j2w})}{1 + a^4 - a^2(e^{j2w} + e^{-j2w})} = 1$$

$$H(z) = \frac{-1}{a^2} + \left(\frac{1}{a^2} - a^2\right) \cdot \frac{1}{1 - a^2 z^{-2}}$$

$$h[n] = -\frac{1}{a^2} \delta[n] + \left(\frac{1}{a^2} - a^2\right) \overbrace{TZ^{-1}\left(\frac{1}{1 - a^2 z^{-2}}\right)}^{H_1(z)}$$

$$H_1(z) = \frac{1}{1 - a^2 z^{-2}} = \frac{A}{1 - a z^{-1}} + \frac{B}{1 + a z^{-1}}$$

$$\rightarrow A = H_1(z)(1 - a z^{-1})\Big|_{a z^{-1} = 1}$$

$$\rightarrow B = H_1(z)(1 + a z^{-1})\Big|_{a z^{-1} = -1}$$

$$h_1[n] = A a^n u[n] + B(-a)^n u[n]$$

Sea $E_x = 1 \rightarrow E_y$??

$$E_x = \sum |x[n]|^2 = 1 = \{\text{Parseval}\} = \frac{1}{2\pi} \int_{2\pi} |X(e^{jw})|^2 \, dw$$

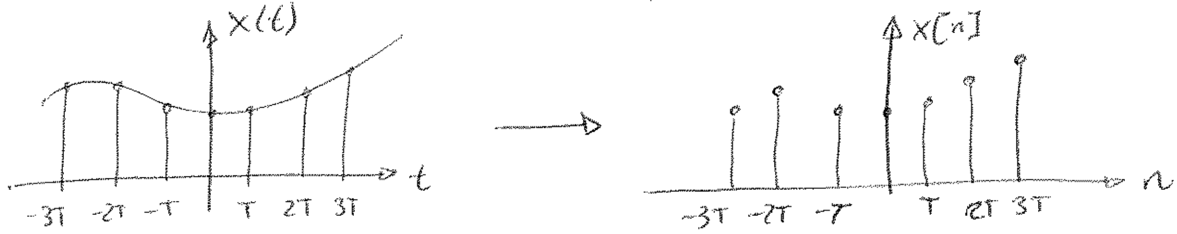
$$E_y = \frac{1}{2\pi} \int_{2\pi} |X(e^{jw}) H(e^{jw})|^2 \, dw = \{|H(e^{jw})| = 1\} = E_x = 1$$

Tema 4: Muestreo

Representación de una señal continua a partir de sus muestras equiespaciadas

$$x(t) \longrightarrow \boxed{\text{muestreo}} \longrightarrow x[n] = x(nT)$$

T = periodo de muestreo
T ≠ periodo de la señal $x(t)$



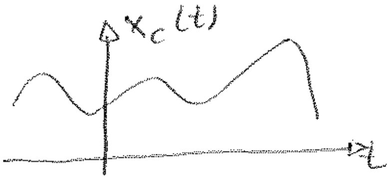
$$x_c(t) \longrightarrow \underset{\uparrow T}{\boxed{C/D}} \xrightarrow{x_d[n]} \underset{\uparrow T}{\boxed{D/C}} \longrightarrow x_r(t)$$

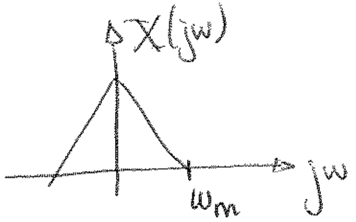
$x_c(t)$ = tiempo continuo
$x_d(t)$ = tiempo discreto
C/D = conversor Continuo/Discreto

$$x_d[n] = x_c(t)\big|_{t=nT} = x_c(nT)$$

- El conversor C/D ó D/C transforma la señal C (tal vez infinita) en tantos valores reales como sea necesario.
- El conversor A/D ó D/A (Analógico/Digital) transforma la señal A (tal vez infinita) en un número finito de valores (pues tenemos memoria limitada) de carácter no real: los valores están cuantificados ("redondeados").

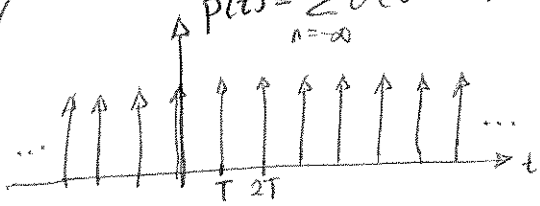
Muestreo ideal:

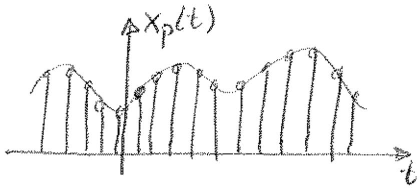




$$x_c(t) \longrightarrow \otimes \longrightarrow x_p(t) = x_c(t)\cdot\sum_{-\infty}^{+\infty}\delta(t-nT)$$

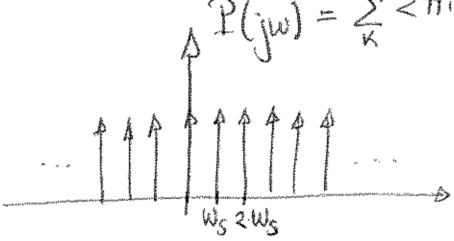
$$p(t) = \sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

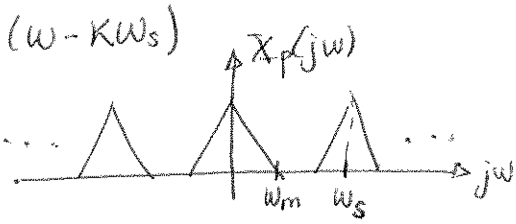




$$X_c(jw) \longrightarrow \circledast \longrightarrow X_p(jw) = \frac{1}{2\pi}X(jw) * P(jw)$$

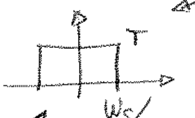
$$P(jw) = \sum_K 2\pi\frac{1}{T}\,\delta(w-Kw_s)$$





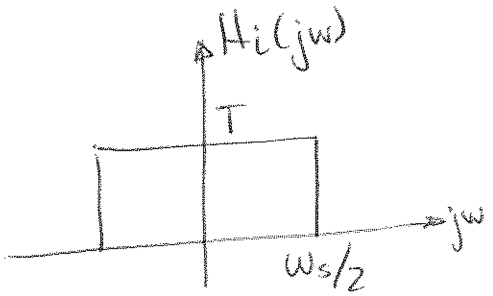
$$x_p(t) = \sum_n x_c(nT)\cdot\delta(t-nT) \longrightarrow x_r(t)\ ?? \rightarrow \text{ni idea}$$

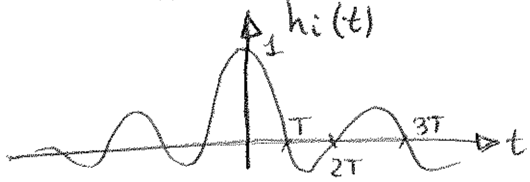
$$X_p(jw) = \frac{1}{T}\sum_K X_c(j(w-Kw_s)) \longrightarrow X_r(jw)??$$

$$X_r(jw) = X_p(jw)\cdot$$  $$\approx X_c(jw)$$

$$H_i(jw) = \begin{cases} T, & |w| < \frac{w_s}{2} \\ 0, & \text{resto} \end{cases}$$

$$h_i(t) = \frac{\text{sen}(t\frac{w_s}{2})\cdot T}{\pi\cdot t} = \frac{\text{sen}(t\,\pi/T)}{t\cdot\pi/T}$$

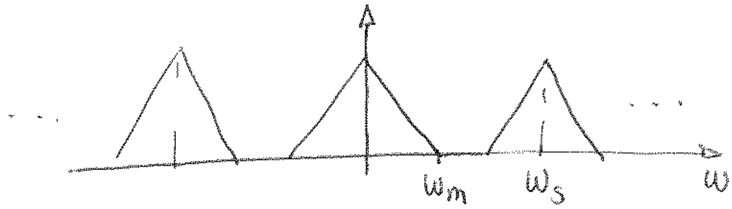




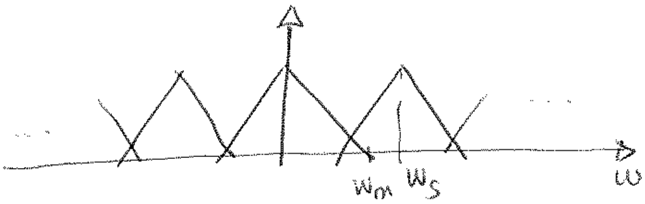
interpolador

$$X_r(jw) = X_c(jw)\cdot H_i(jw) \longleftrightarrow x_r(t) = x_c(t) * h_i(t)$$

Sólo podremos recuperar la señal $x_c(t)$, $X_c(jw)$ si la señal muestreada no se solapa en frecuencia:



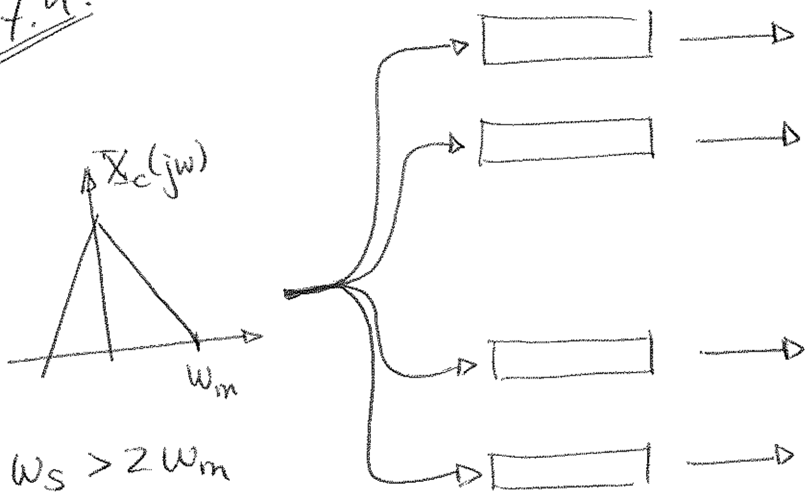
recuperable, ya que $w_s \geq 2w_m$



No recuperable !!

Sólo es recuperable si $\boxed{w_s \geq 2w_m}$ T^{ma} Nyquist

Ej 7.4:



$y_1(t) = x(t) + x(t-1)$

$y_2(t) = \dfrac{dx(t)}{dt}$

$y_3(t) = x^2(t)$

$y_4(t) = x(t) \cdot \cos(w_s t)$

$w_s > 2w_m$

$X_c(jw) \longrightarrow \otimes \longrightarrow X_p(jw) = \frac{1}{T}\sum X(j(w-kw_s))$

$p(t) = \sum_n \delta(t-nT)$

$P(jw) = \sum_k 2\pi \frac{1}{T}\delta(w-kw_s)$

$$X_p(jw) = \frac{1}{2\pi} X(jw) * P(jw) =$$

$$= \frac{1}{\cancel{2\pi}} \sum_k \cancel{2\pi} \frac{1}{T} \delta(w-kw_s) * X(jw) =$$

$$= \frac{1}{T} \sum_k X(j(w-kw_s))$$

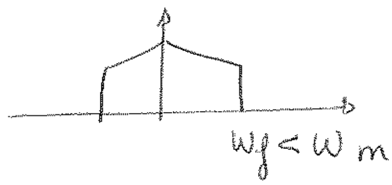
a) $y(t) = x(t) + x(t-1)$

$x(t) \longrightarrow \boxed{\begin{matrix} h(t) = \delta(t) + \delta(t-1) \\ H(jw) = 1 + e^{-jw} \end{matrix}} \xrightarrow{y(t) = x(t) * h(t)} \otimes \longrightarrow y(t) \cdot p(t)$

$\uparrow p(t)$

$w_s \geq 2w_m$

si $H(jw)$ fuese:

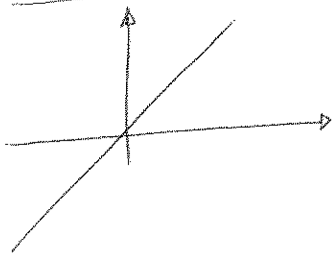


$\Longrightarrow$ condición de recuperabilidad $w_s \geq 2w_f$

Luego: $w_s \geq 2 \cdot \min(w_m, w_f)$

b) $y(t) = \dfrac{dx(t)}{dt}$

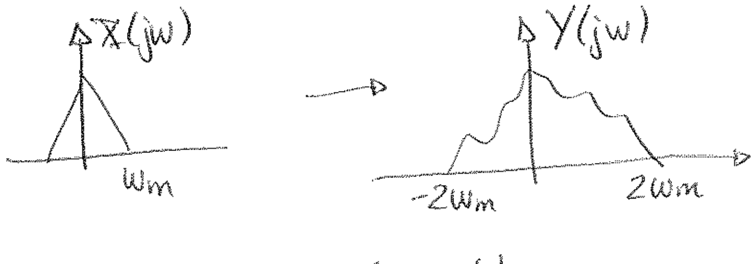
$\longrightarrow \boxed{H(jw) = jw} \longrightarrow Y(jw) = X(jw) \cdot H(jw)$



$w_s \geq 2w_m$

c) $y(t) = x^2(t) \Longrightarrow Y(jw) = \frac{1}{2\pi} X(jw) * X(jw)$

no es lineal $\Rightarrow$ no usaremos $h(t)$ ni $H(jw)$

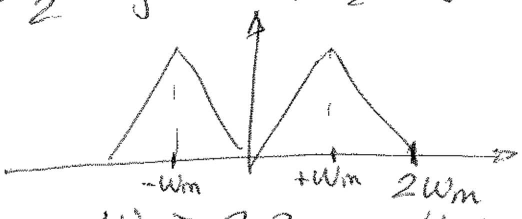


$w_s \geq 2 \cdot (2w_m) = 4w_m$

d) $y(t) = x(t) \cdot \cos(w_m t)$

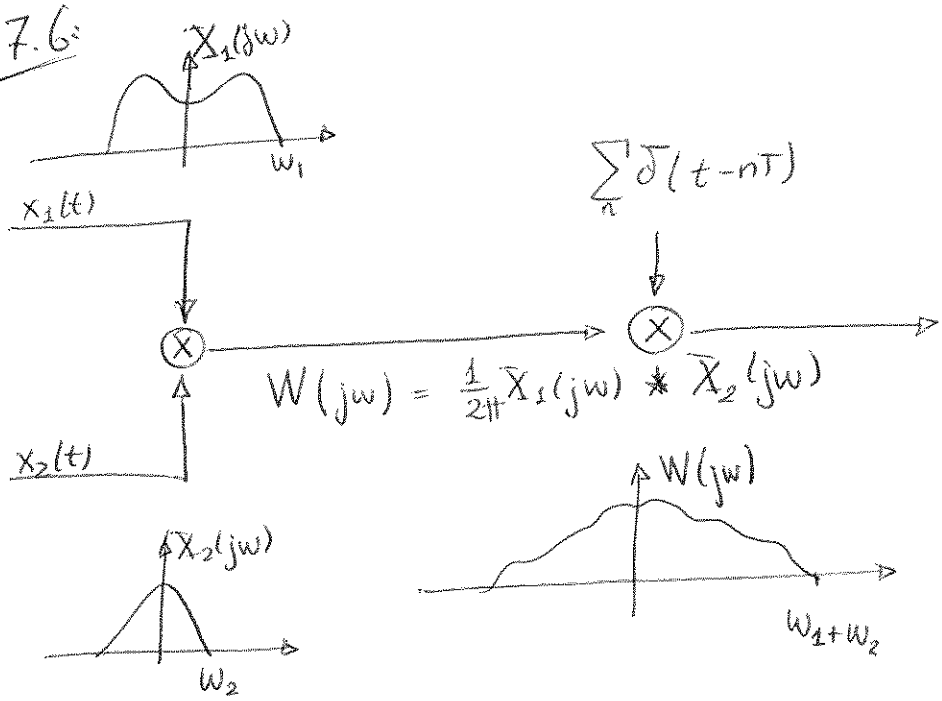
$x(t) \longrightarrow \otimes \longrightarrow Y(jw) = \frac{1}{2} X(j(w - w_m)) + \frac{1}{2} X(j(w + w_m))$

$\uparrow \cos(w_m t)$

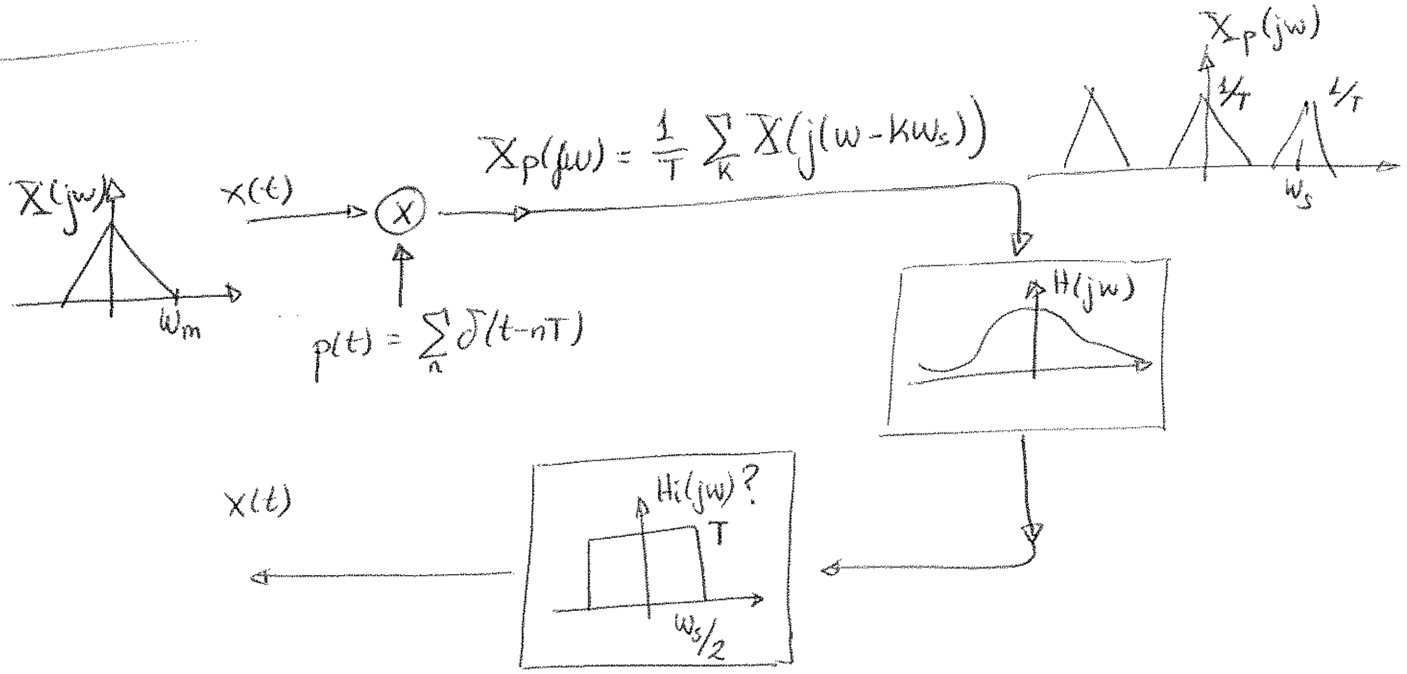


$w_s \geq 2 \cdot 2w_m = 4w_m$

Ej 7.6:



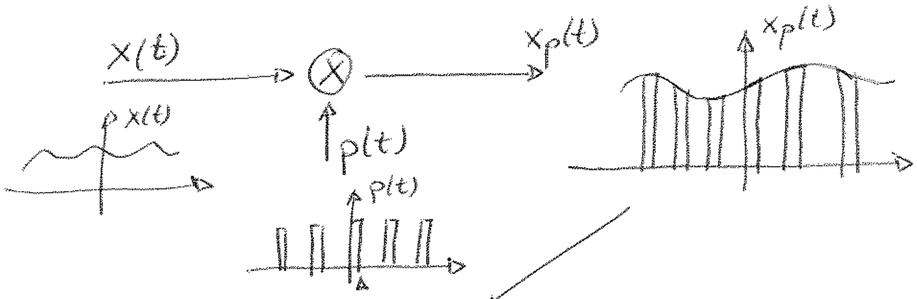
$$W_s \geq 2(w_1+w_2)$$



para obtener $x(t)$ a la salida necesitamos que:

$$H(jw)\cdot H_i(jw) = \begin{cases} T, & |w| < \frac{W_s}{2} \\ 0, & \text{resto} \end{cases} \quad ; \quad H_i(jw) = \begin{cases} T/H(jw), & |w| < \frac{W_s}{2} \\ 0 & \text{resto} \end{cases}$$

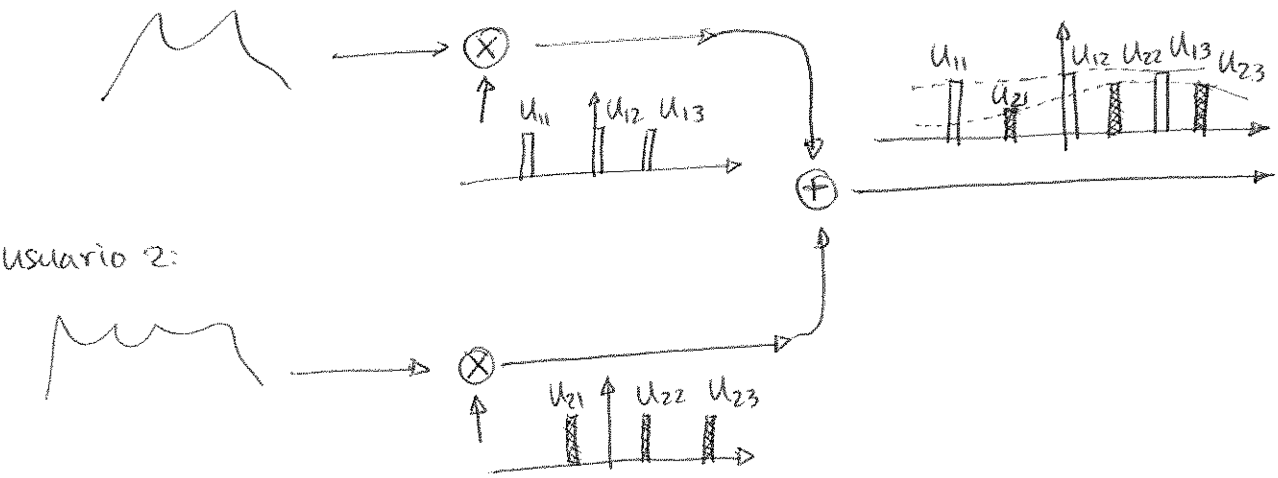
Modulación por pulsos:



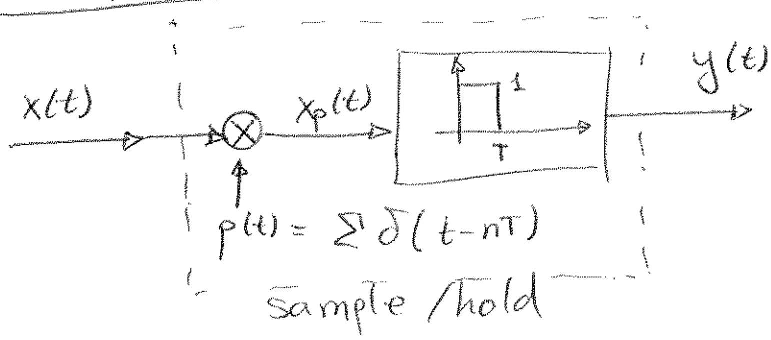
$$X_p(j\omega) = \sum a_k X(j(\omega - k\omega_s)) \; ; \; a_k = \left.\frac{P_p(j\omega)}{T}\right|_{\omega = k\omega_s}$$

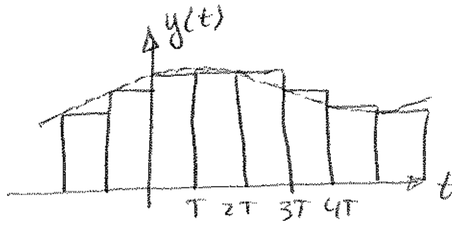
$$P_p(j\omega) = \frac{2\,sen(\omega\Delta/2)}{\omega}\, e^{-j\omega\frac{\Delta}{2}}$$

En una TDMA: Time Division Multiplexation A.

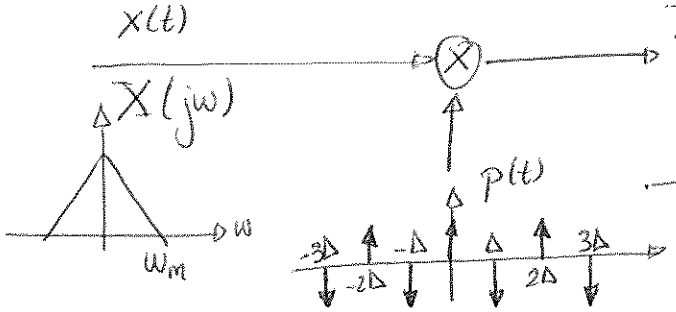


Sample / hold : muestrea / retiene



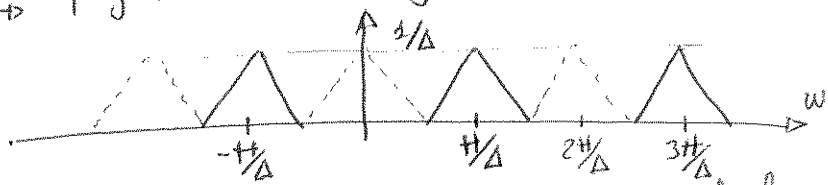


Ej 7.23:



$$X_p(jw) = \frac{1}{2\pi} X(jw) * P(jw)$$

$$X_p(jw) = \sum a_K \cdot X(j(w - Kw_s))$$



condición recuperabilidad: $\frac{\pi}{\Delta} \geq w_m$

$$P(jw) = \sum 2\pi a_K \delta(w - Kw_s)$$

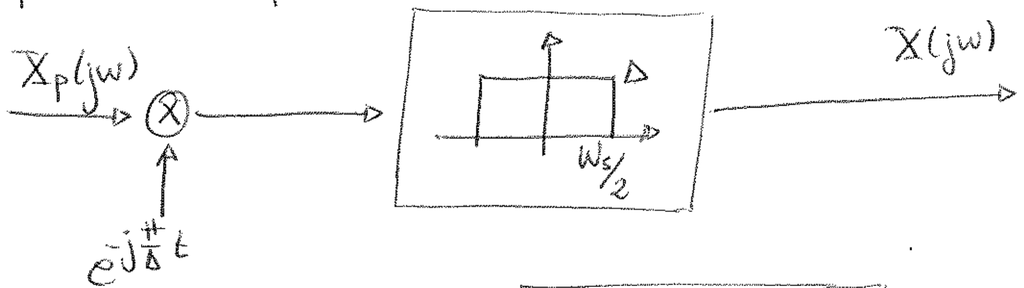
$$w_s = \frac{2\pi}{T} = \frac{2\pi}{2\Delta} = \frac{\pi}{\Delta}$$

$$p_p(t) = \delta(t) - \delta(t-\Delta)$$

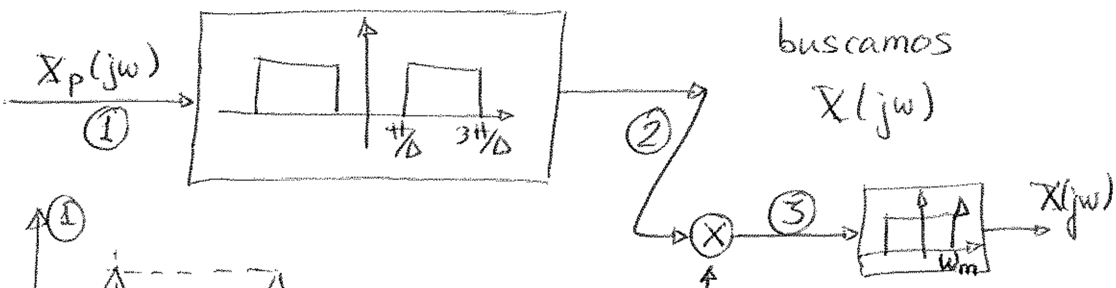
$$P_p(jw) = 1 - e^{-jw\Delta}$$

$$a_K = \frac{P_p(jw)}{T}\bigg|_{w=Kw_s} = \frac{1 - e^{-jK\frac{\pi}{\cancel{\Delta}}\cancel{\Delta}}}{2\Delta} = \frac{1-(-1)^K}{2\Delta} \begin{cases} K=\text{par} \rightarrow = 0 \\ K=\text{impar} \rightarrow = 1/\Delta \end{cases}$$

para recuperar $x(t)$:



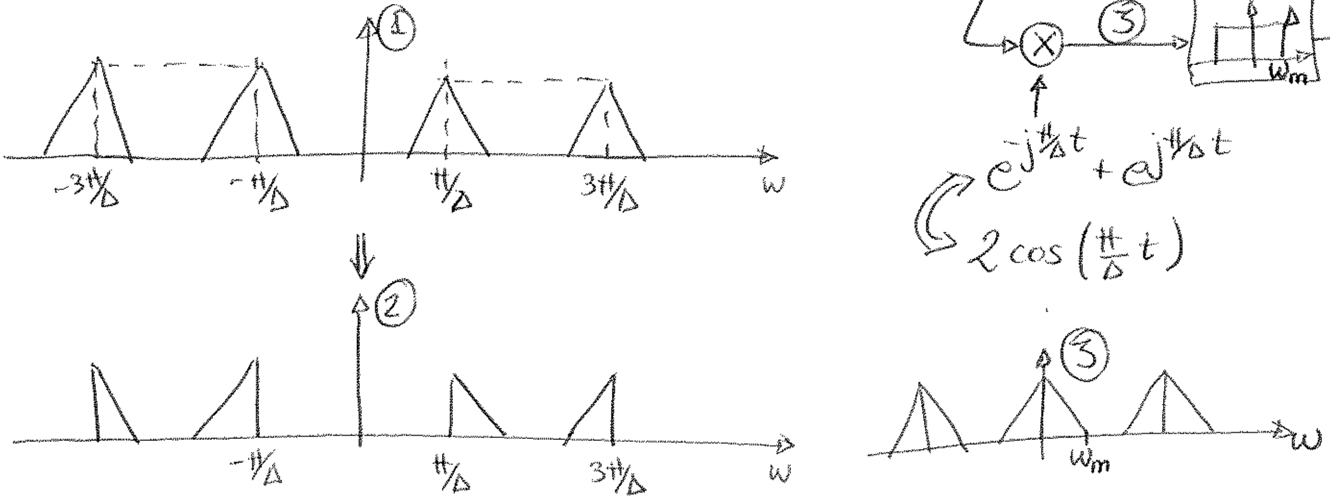
b) si tenemos:



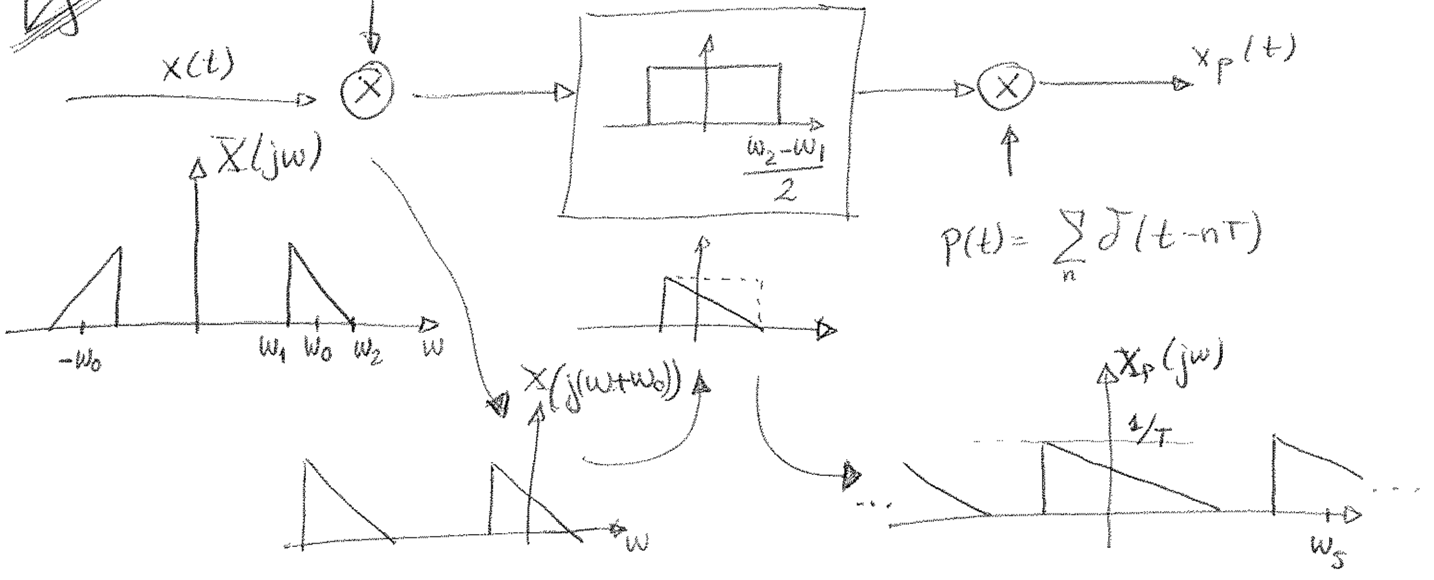
buscamos $X(jw)$

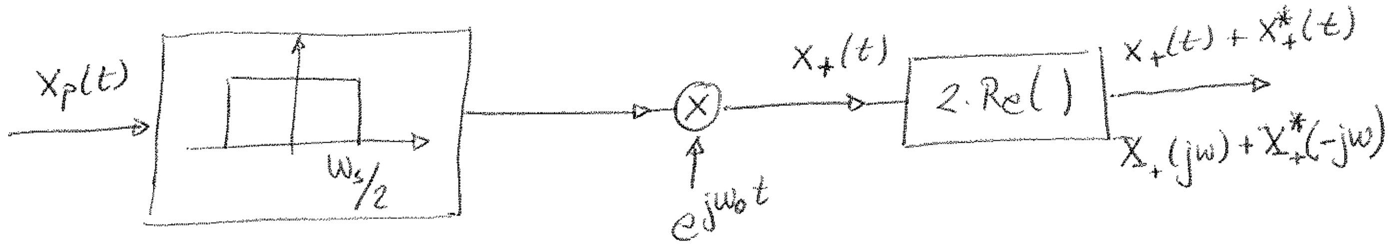
$$e^{-j\frac{\pi}{\Delta}t} + e^{j\frac{\pi}{\Delta}t}$$

$$2\cos\left(\frac{\pi}{\Delta}t\right)$$



Ej 7.27

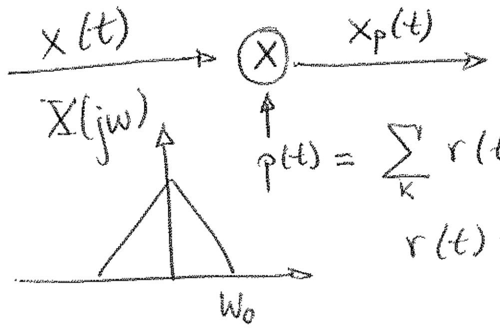




condición:

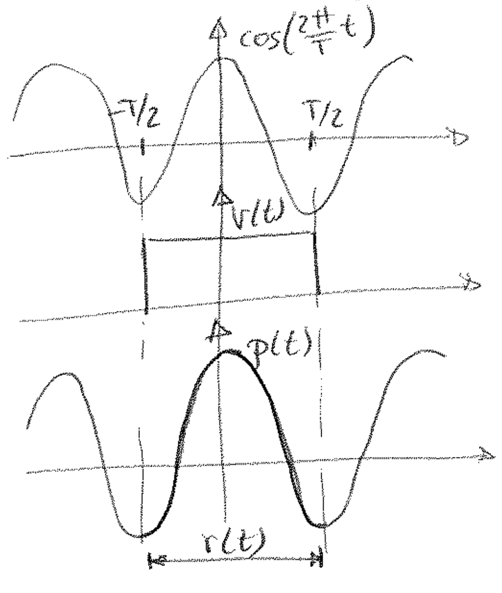
$$\omega_s \geq 2\frac{(\omega_2-\omega_1)}{2} = \omega_2 - \omega_1$$

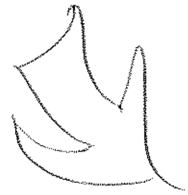
sep06



$$p(t) = \sum_k r(t-kT) = \cos\left(\frac{2\pi}{T}t\right)$$

$$r(t) = \cos\left(\frac{2\pi}{T}t\right)\cdot\left[u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)\right]$$





$$r(t) = \cos\left(\frac{2\pi}{T}t\right)\cdot v(t)$$

$$R(jw) = \frac{1}{\cancel{2\pi}}\left(\cancel{\pi}\,\delta\left(w-\frac{2\pi}{T}\right) + \cancel{\pi}\,\delta\left(w+\frac{2\pi}{T}\right)\right) * \frac{\cancel{2}\,\text{sen}(wT/2)}{w} =$$

$$= \frac{\text{sen}\left(\left(w-\frac{2\pi}{T}\right)\frac{T}{2}\right)}{w-2\pi/T} + \frac{\text{sen}\left(\left(w+\frac{2\pi}{T}\right)\frac{T}{2}\right)}{w+2\pi/T}$$

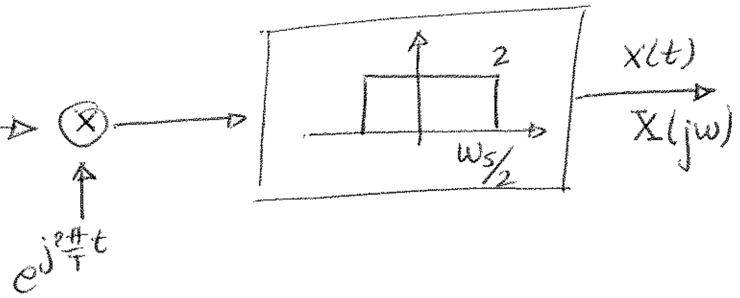
$$P(jw) = \sum 2\pi\, a_k\, \delta\left(w - k\frac{2\pi}{T}\right)$$

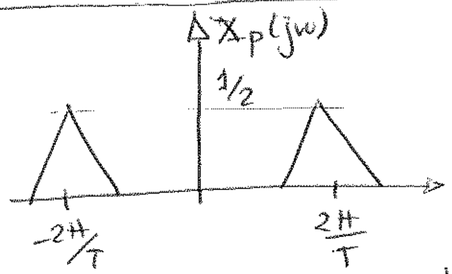
$$a_k = \left.\frac{R(jw)}{T}\right|_{w=\frac{2k\pi}{T}} = \frac{\text{sen}\left(\left(k\frac{2\pi}{T}-\frac{2\pi}{T}\right)\frac{T}{2}\right)}{T\left(k\frac{2\pi}{T}-\frac{2\pi}{T}\right)} + \frac{\text{sen}\left(\left(k\frac{2\pi}{T}+\frac{2\pi}{T}\right)\frac{T}{2}\right)}{T\left(k\frac{2\pi}{T}+\frac{2\pi}{T}\right)}$$

$$= \frac{\text{sen}((k-1)\pi)}{(k-1)2\pi} + \frac{\text{sen}((k+1)\pi)}{(k+1)2\pi} = \begin{cases} a_1 = a_{-1} = 1/2 \\ a_k = 0 \quad \forall k \neq 1,-1 \end{cases}$$

$$P(jw) = \pi\,\delta\left(w-\frac{2\pi}{T}\right) + \pi\,\delta\left(w+\frac{2\pi}{T}\right)$$

$$X_p(jw) = \frac{1}{2\pi} X(jw) * P(jw)$$

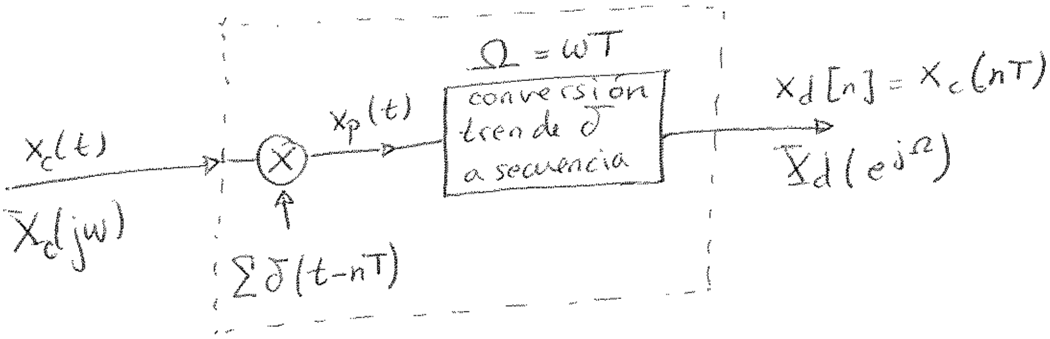




condición de recuperabilidad:

$$w_0 < \frac{2\pi}{T}$$

Bloque C→D: Continuo → Discreto



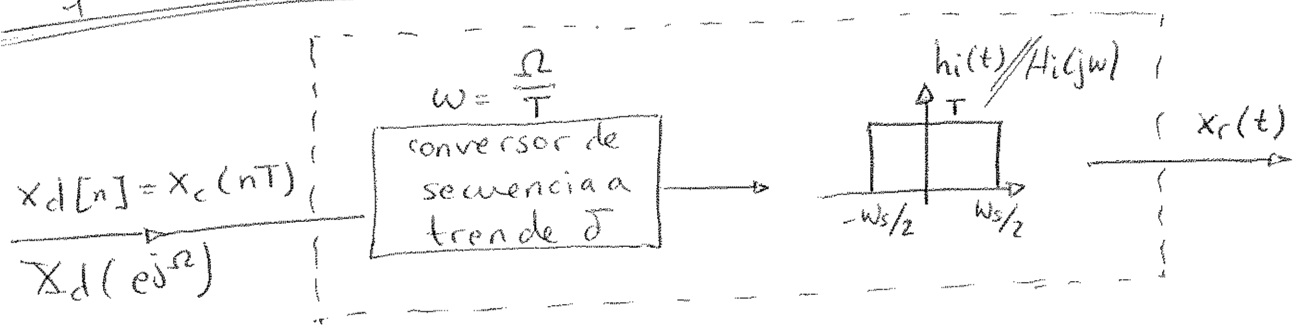
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\,\delta(t-nT)$$

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} X_c(j(\omega - k\omega_s)) = \sum_{n=-\infty}^{+\infty} x_c(nT)\, e^{-j\omega nT} = \sum x_c(nT)\, e^{-j\Omega n}$$

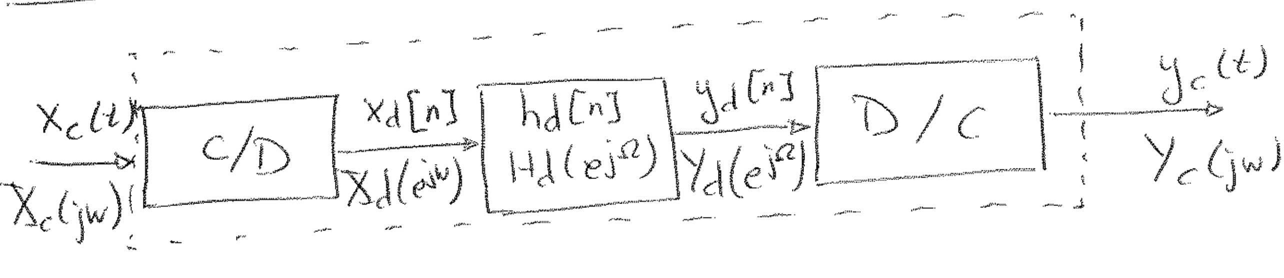
$$X_d(e^{j\Omega}) = \sum x_d[n]\, e^{-j\Omega\omega} = \sum x_c(nT)\, e^{-j\Omega\omega}$$

$$X_d(e^{j\Omega}) = \sum \frac{1}{T} X_c\left(j\frac{\Omega - 2k\pi}{T}\right)$$

Bloque D→C: Discreto → Continuo



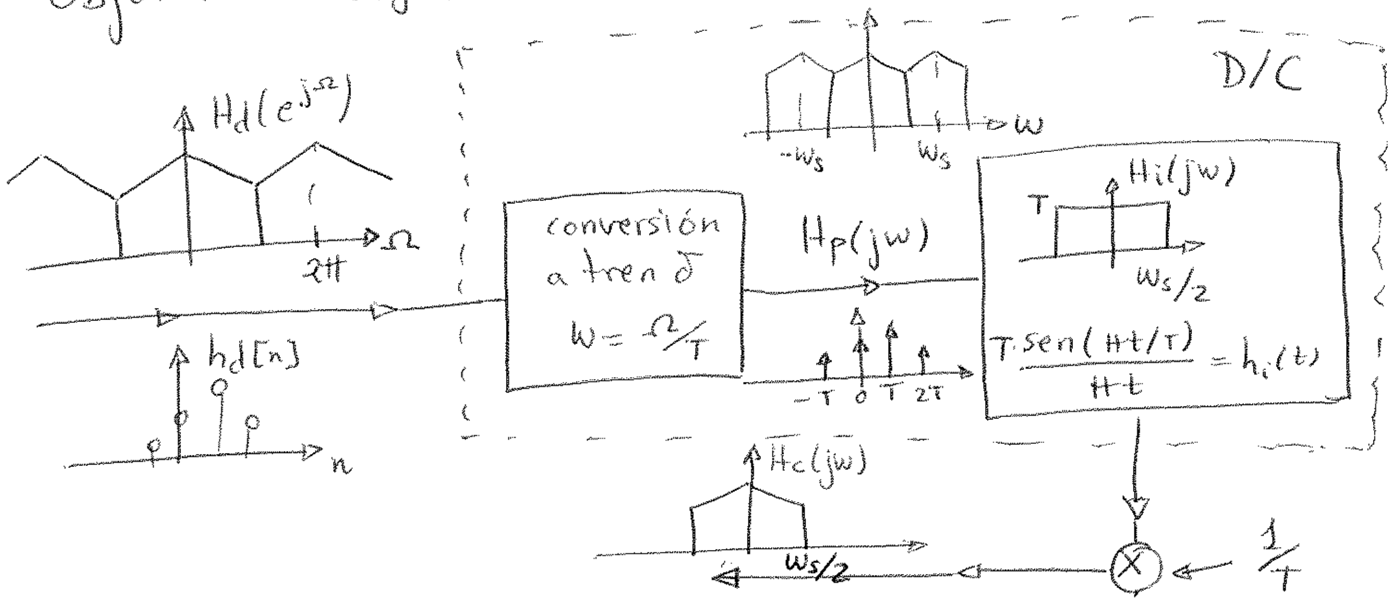
Sistema:



Planteamiento 1:

Datos: $H_d(e^{j\Omega}) \longleftrightarrow h_d[n]$

Objetivo: $H_c(j\omega) \longleftrightarrow h_c(t)$



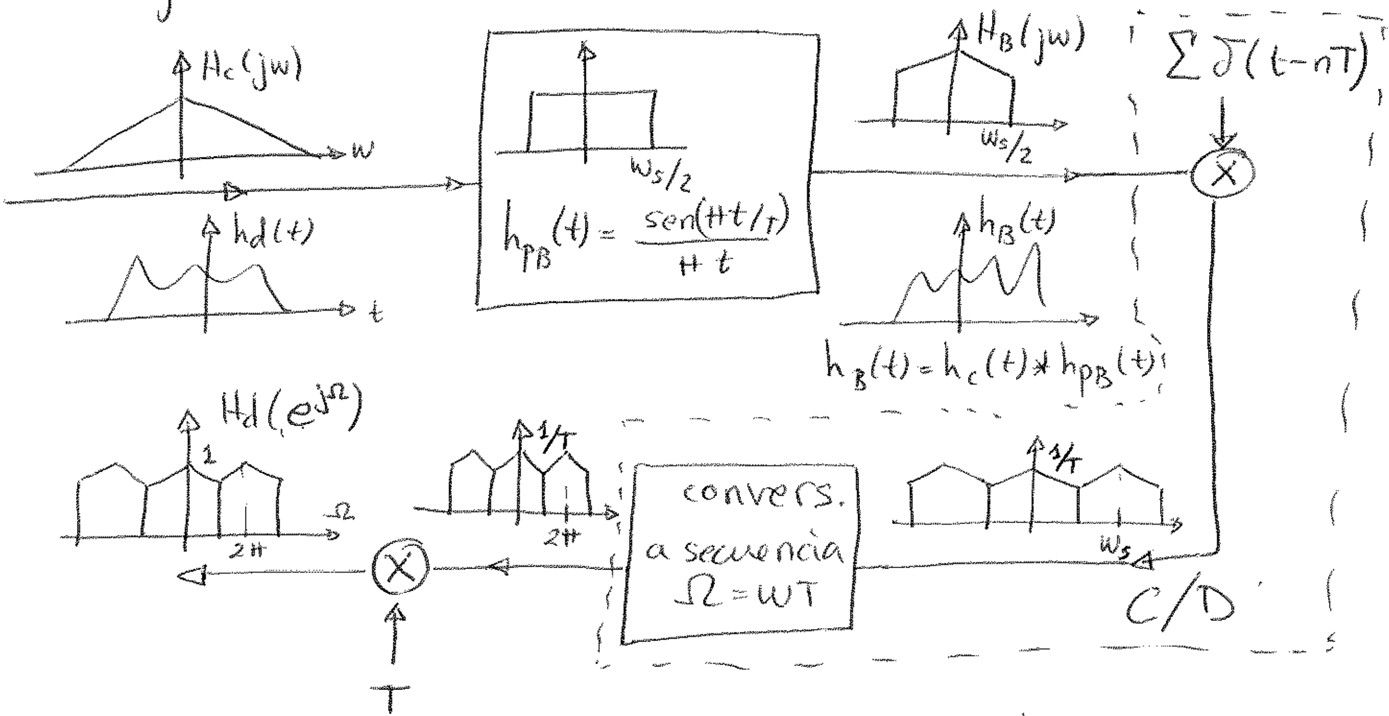
$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{resto} \end{cases}$$

$$h_c(t) = \sum \frac{1}{T} h_d[n] \cdot h_i(t - nT)$$

Planteamiento 2:

Datos: $H_c(j\omega) \longleftrightarrow h_c(t)$

Objetivo: $H_d(e^{j\Omega}) \longleftrightarrow h_d[n]$



$$H_d(e^{j\omega}) = H_c\left(j\frac{\Omega}{T}\right), \quad |\Omega| < \pi \text{ (periódica } 2\pi\text{)}; \quad h_d[n] = T\, h_B(nT)$$

sep 98

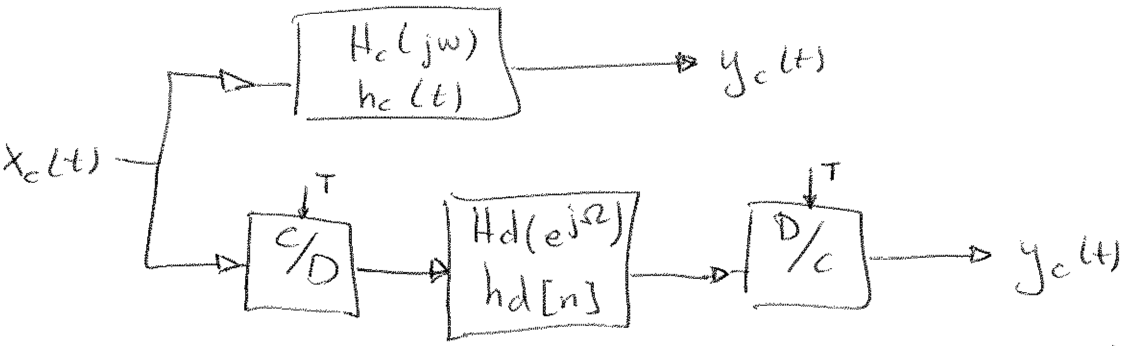
$$y_c(t) = x_c(t) + \alpha\, x_c(t - t_0)$$

a) $h_c(t)$?? $H_c(jw)$??

$$h_c(t) = y_c(t)\Big|_{x_c(t) = \delta(t)} = \delta(t) + \alpha\, \delta(t - t_0)$$

$$H_c(jw) = \mathcal{F}(h_c(t)) = 1 + \alpha\, e^{-jwt_0}$$

b)



$$H_c(jw) \longrightarrow H_d(e^{j\Omega}) = \begin{cases} H_c\left(j\frac{\Omega}{T}\right), & |\Omega| < \pi \\ \text{periódica } 2\pi \end{cases} = \begin{cases} 1 + \alpha\, e^{-j\frac{\Omega}{T}t_0}, & |\Omega| < \pi \\ \text{periódica } 2\pi \end{cases}$$

siendo $t_0 = kT$ (enunciado):

$$h_d[n] = \delta[n] + \alpha\, \delta[n - k]$$

si $t_0 \neq kT$:

$$h_d[n] = \delta[n] + \alpha \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\frac{\Omega}{T}t_0}\, e^{j\Omega n}\, d\Omega = \delta[n] + \frac{\alpha\left(e^{j\pi(n - \frac{t_0}{T})} - e^{-j\pi(n - \frac{t_0}{T})}\right)}{2\pi j(n - \frac{t_0}{T})} = \delta[n] + \alpha\, \frac{\operatorname{sen}(\pi(n - \frac{t_0}{T}))}{\pi(n - \frac{t_0}{T})} =$$

$$= \left\{ \text{si } \frac{t_0}{T} = k \right\} = \delta[n] + \alpha\, \delta[n - k]$$

$h_c(t) = \delta(t) + \alpha\,\delta(t-t_0) \longrightarrow \boxed{h_{PB}(t) = \dfrac{sen(\pi t/T)}{\pi t}} \xrightarrow{h_B(t)} \boxed{C/D} \to \otimes \leftarrow T$

$$h_B(t) = \frac{sen(\pi t/T)}{\pi t} + \alpha\,\frac{sen(\pi(t-t_0)/T)}{\pi(t-t_0)}$$

$h_d[n] = T\cdot h_B(nT)$

$t = nT$

$$h_d[n] = T\left(\frac{sen(\pi nT/T)}{\pi nT}\right) + \alpha T\,\frac{sen(\pi(nT-t_0)/T)}{\pi(nT-t_0)} =$$

$$= \delta[n] + \alpha\,\frac{sen(\pi(n-t_0/T))}{\pi(n-t_0/T)}$$

Ej 7.30

$x(t) \longrightarrow \boxed{\dfrac{dy_c(t)}{dt} + y_c(t) = x_c(t) \quad \text{LTI causal}} \xrightarrow{y_c(t)}$

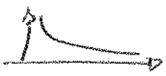
a) $Y_c(s)\cdot s + Y_c(s) = X_c(s) \Rightarrow H_c(s) = \dfrac{Y_c(s)}{X_c(s)} = \dfrac{1}{1+s}$

b) $H_d(e^{j\Omega})$?? $h_d[n]$?? $\quad w[n] = \delta[n]$

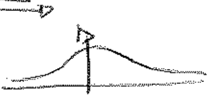
$x_c(t) = \delta(t) \longrightarrow \boxed{ED}$ (LTI) $\xrightarrow{y_c(t) = e^{-t}\cdot u(t)}$ A $\to \otimes \to$ B $\to$ $\boxed{\text{conversión a secuencia}}$ $\to$ C $\xrightarrow{y[n]}$ $\boxed{H_d(e^{j\Omega}) \quad h_d[n]}$ $\to W[n] = \delta[n]$

$y_c(t) = h_c(t)$ $\qquad \sum \delta(t-nT)$ $\qquad y_p(t)$

A) $y_c(t)$:

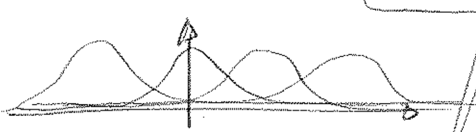


$Y_c(j\omega) = \dfrac{1}{1+j\omega}$



B) $y_p(t) = \sum_n y_c(nT)\cdot\delta(t-nT)$

$Y_p(j\omega) = \dfrac{1}{T}\sum_k X_c(j(\omega - k\omega_s))$



C) $y_d[n] = y_c(nT) = e^{-nT}u[n]$

$Y_d(e^{j\Omega}) = Y_p(j\omega)\Big|_{\Omega=\omega T} = \dfrac{1}{T}\sum X_c\left(j\,\dfrac{\Omega - k2\pi}{T}\right)$

Para que $w[n] = \delta[n]$

$H_d(e^{j\Omega})\cdot Y_d(e^{j\Omega}) = 1$

$H_d(e^{j\Omega}) = \dfrac{1}{Y_d(e^{j\Omega})}$

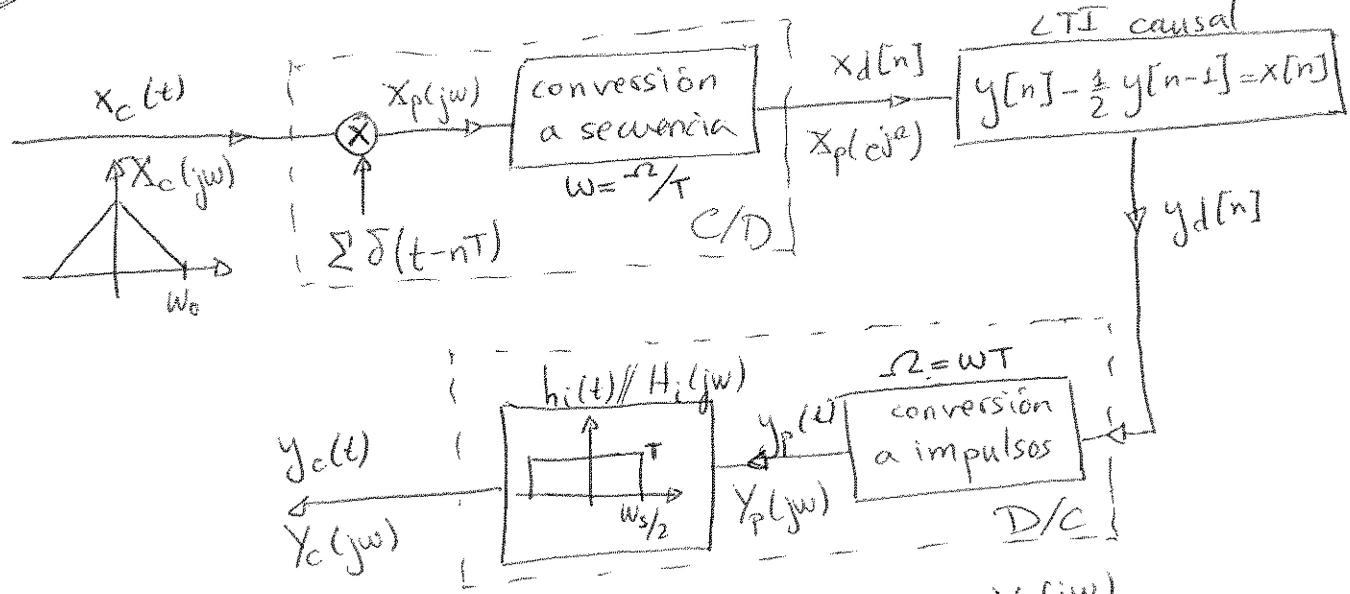
como la expresión es demasiado fea para hallar $h_d[n]$ a partir de $H_d(e^{j\Omega})$, realizamos:

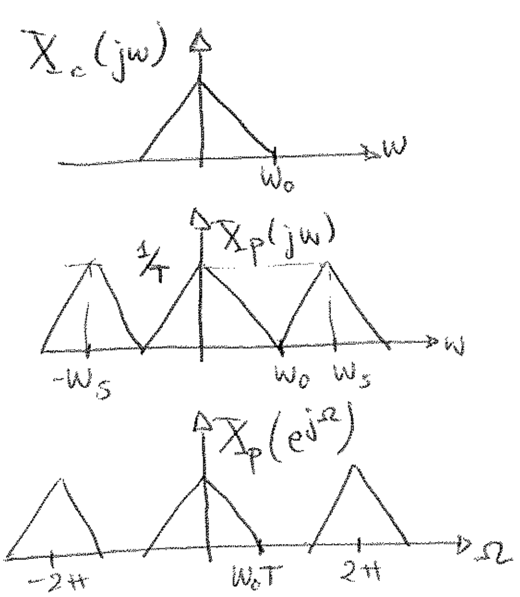
$$Y_d(e^{j\Omega}) = TF(y_d[n]) = TF((e^{-T})^n u[n]) = \frac{1}{1 - e^{-T}e^{-j\Omega}}$$

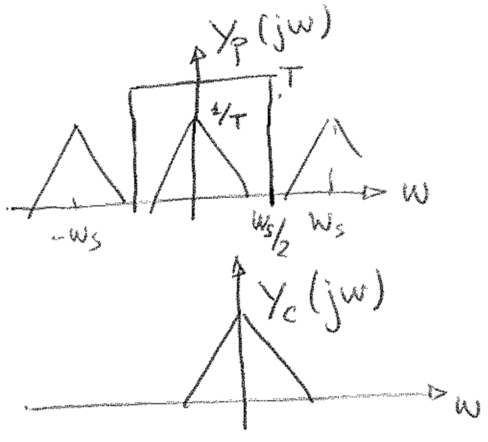
$$H_d(e^{j\Omega}) = \frac{1}{Y_d(e^{j\Omega})} = 1 - e^{-T}e^{-j\Omega}$$

$$h_d[n] = \delta[n] - e^{-T}\delta[n-1]$$

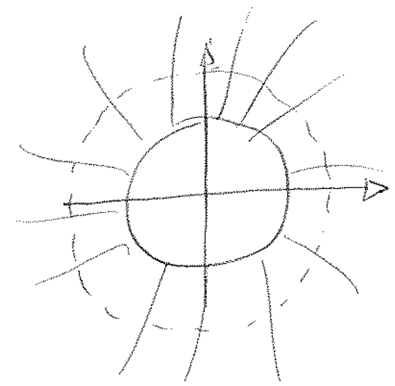
Ej 7.31







$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}$$



$$H(e^{j\Omega}) = H(z)\Big|_{z=e^{j\Omega}} = \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$H_c(jw) = \begin{cases} H_d(e^{j\Omega})\Big|_{\Omega = wT}, & |w| < w_s/2 \\ 0 & \text{resto} \end{cases}$$

$$h_c(t) = \sum h_d[n] \cdot \frac{\operatorname{sen} \pi(t-nT)/T}{\pi(t-nT)}$$

$$H_c(jw) = \begin{cases} \dfrac{1}{1-\frac{1}{2}e^{-jwT}}, & |w| < \dfrac{w_s}{2} \\ 0 & \text{resto} \end{cases}$$

$$h_c(t) = TF^{-1}\left(H_c(jw)\right)$$

$$h_c(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{\operatorname{sen} \pi(t-nT)/T}{\pi(t-nT)}$$

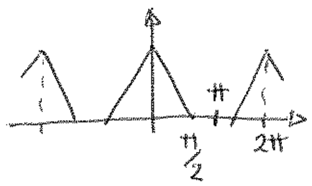
<u>Sep 2000</u>

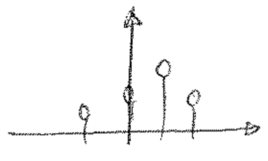
$x[n]$ → conversor a tren imp. → $z(t) = \sum x[k]\,\delta(t-kT)$

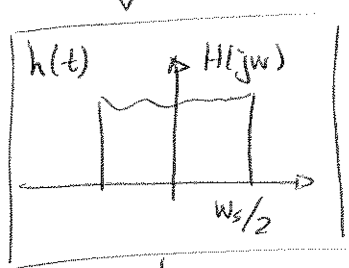
$$h_p(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{resto} \end{cases}$$

$$H_p(jw) = \frac{2\operatorname{sen}(wT/2)}{w} e^{jwT/2}$$

↓ $w(t)$







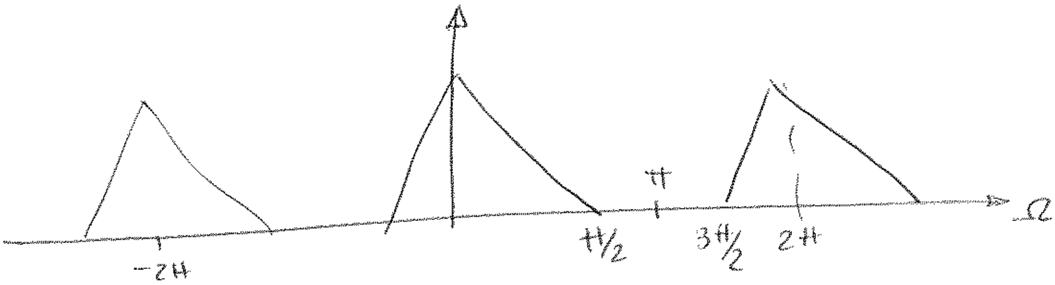
$$\left.\begin{aligned} h_{eq}(t) &= h_p(t) * h(t) \\ H_{eq}(jw) &= H_p(jw) \cdot H(jw) \end{aligned}\right\}$$

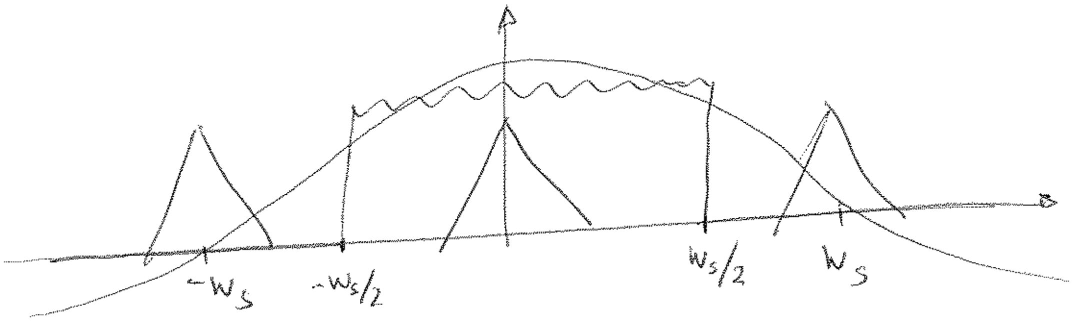
↓ $v(t)$

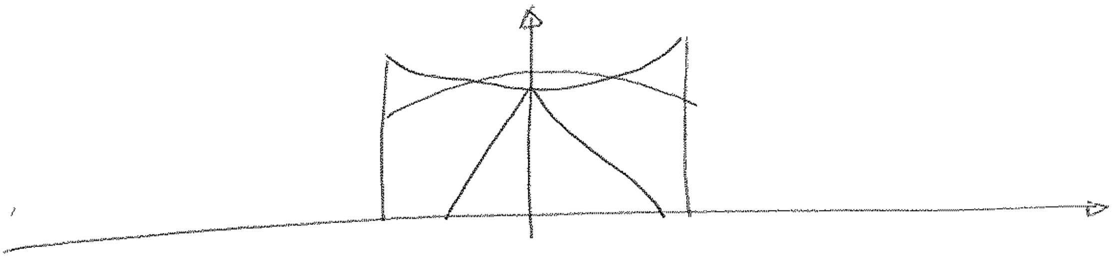
C/D

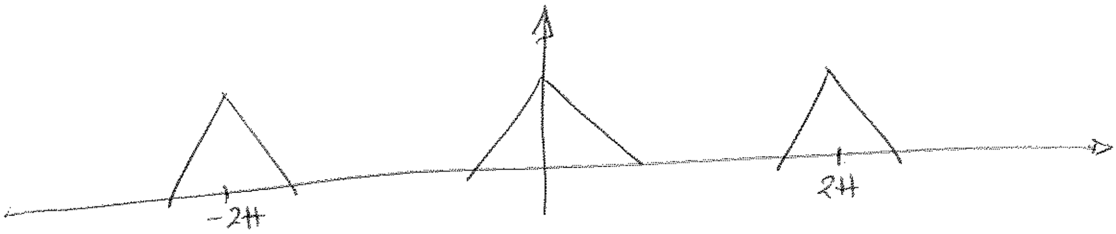
↓ $y[n]$

$$\left.\begin{aligned} y[n] &= x[n] * h_d[n] = \sum_{-\infty}^{+\infty} x[k]\, h_d[n-k] \\ z(t) &= \sum x[k]\, \delta(t-kT) \\ v(t) &= \sum x[k]\, h_{eq}(t-kT) \\ y[n] &= v(nT) = \sum x[k]\, h_{eq}(nT-kT) \end{aligned}\right\} h_d[m] = h_{eq}[mT]$$





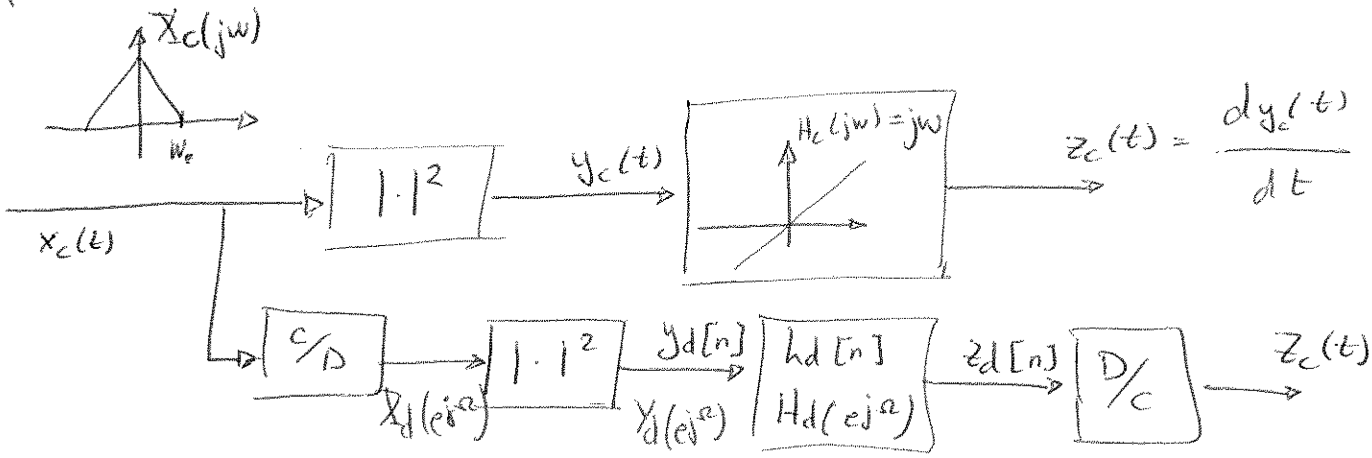




$$H_p(jw)\cdot H(jw) = \begin{cases} T & , |w| < w_s/2 \\ 0 & , \text{resto} \end{cases}$$

sabiendo $H_p(jw) \Rightarrow H(jw) = \begin{cases} \dfrac{wTe^{-jwT/2}}{2\,\text{sen}(wT/2)} & , |w| < w_s/2 \\ 0 & , \text{resto} \end{cases}$

Sep 2002

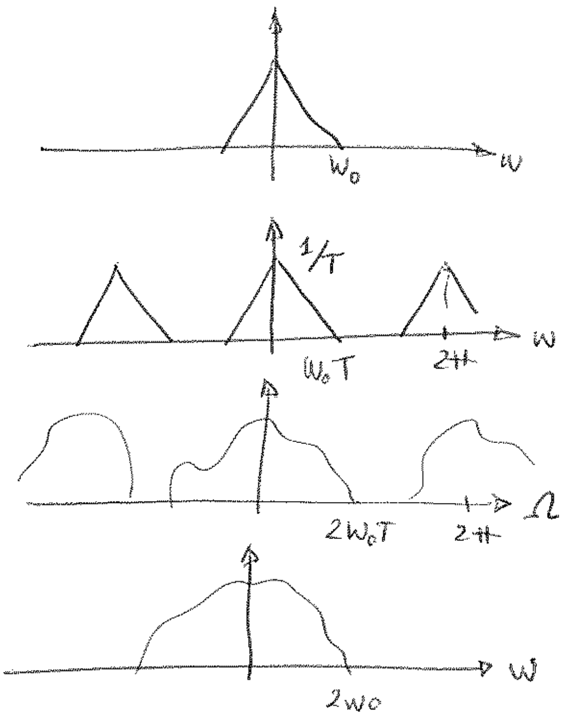


$$Y_d(e^{j\Omega}) = \frac{1}{2\pi} X_d(e^{j\Omega}) \circledast X_d(e^{j\Omega})$$

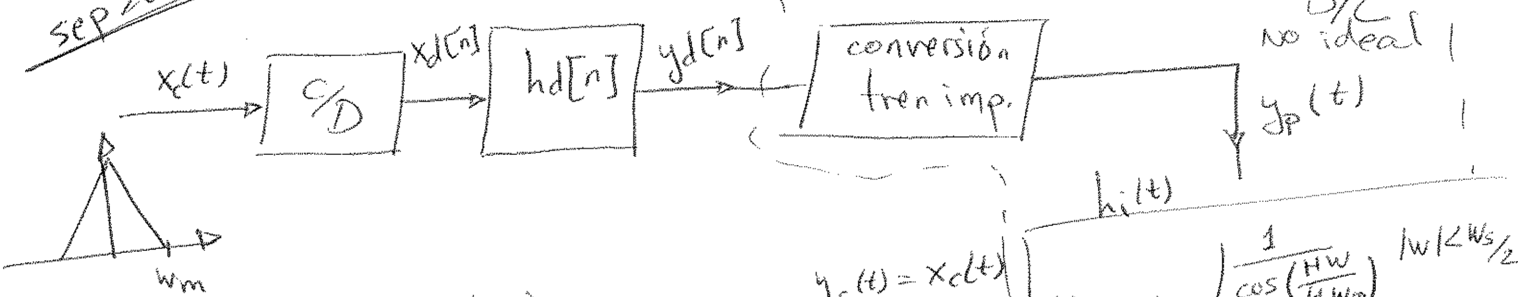
$$H_d(e^{j\Omega}) = \begin{cases} H_c(jw)\big|_{\Omega = wT}, & |\Omega| < \pi \\ \text{periódico } 2\pi \end{cases}$$

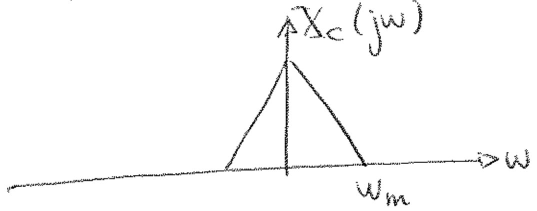
$$H_c(jw)\big|_{\Omega = wT} = j\frac{\Omega}{T}, \quad |\Omega| < \pi$$

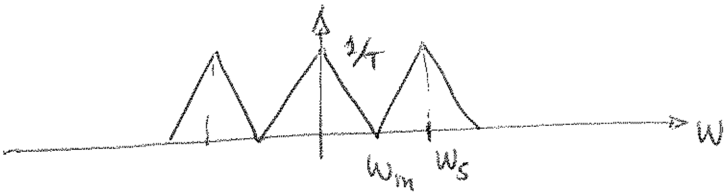
$$h_d[n] = \frac{1}{2\pi} \frac{j}{T} \int_{-\pi}^{\pi} \Omega e^{j\Omega n} d\Omega$$

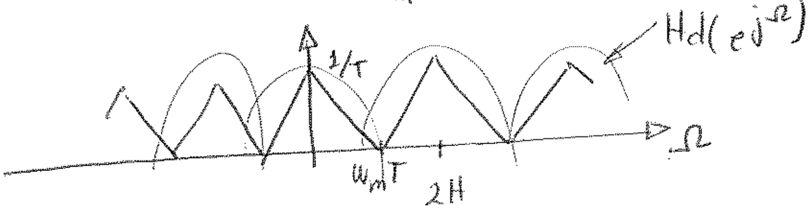


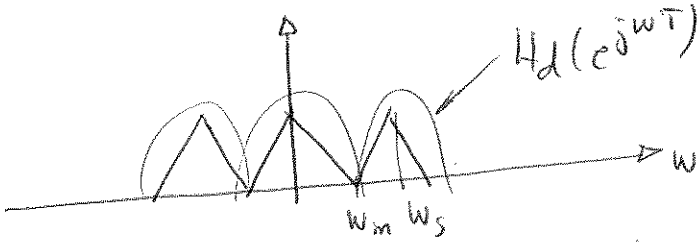
Sep 2009

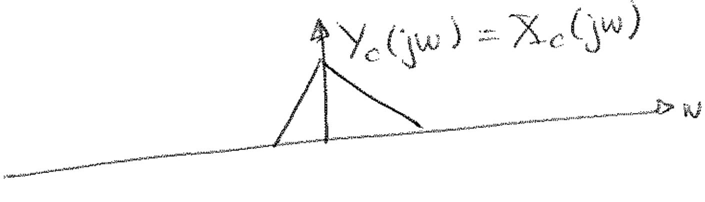












$$H_d(e^{jwT}) \cdot H_i(jw) = \begin{cases} T, & |w| < w_s/2 \\ 0, & \text{resto} \end{cases}$$

$$H_d(e^{j\Omega}) \cdot H_i\left(j\frac{\Omega}{T}\right) = T, \quad |\Omega| < \pi \quad \text{periód. } 2\pi$$

$$H_d(e^{j\Omega}) = \frac{T}{H_i\left(j\frac{\Omega}{T}\right)}, \quad |\Omega| < \pi, \quad \text{periód. } 2\pi$$

$$= T \cos\left(\frac{\pi}{4 w_m} \cdot \frac{\Omega}{T}\right) = T \cdot \cos\left(\frac{\Omega}{4}\right), \quad |\Omega| < \pi$$

$$\Leftrightarrow 4 w_m = 2 w_s \rightarrow \left[w_m = \frac{w_s}{2} \right]$$

$$\Leftrightarrow 2 w_s T = 2 \cdot 2\pi$$

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\Omega}) e^{j\Omega n} \, d\Omega$$

feb06 1

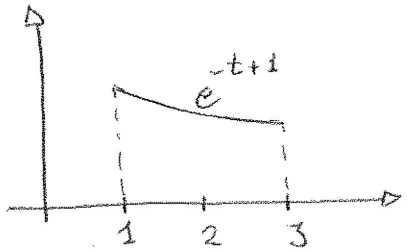
$$x(t) \longrightarrow \boxed{h(t) = x^*(-t)} \longrightarrow y(t) = x(t) * h(t) =$$

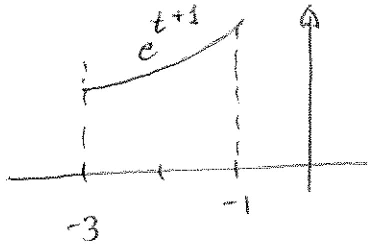
$$y(t) = \int x(\tau)\, h(t-\tau)\, d\tau = \int x(\tau)\, x^*(-(t-\tau))\, d\tau$$

a) $y(0)? \Rightarrow y(0) = \int |x(\tau)|^2 d\tau = E_x$

b) $x(t) = e^{-t+1}\left[u(t-1) - u(t-3)\right]$

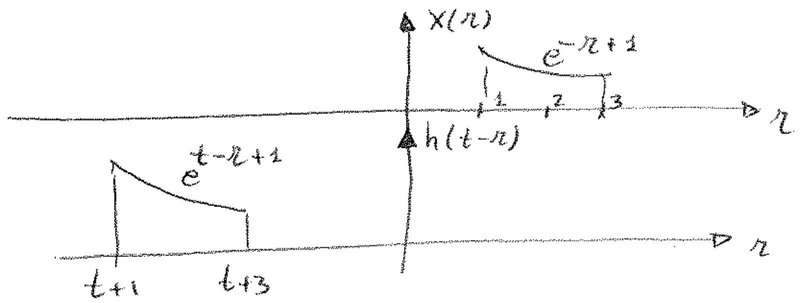
$$h(t) = x(-t) = e^{t+1}\left(u(\underbrace{-t-1}_{\substack{-t>1 \\ t<-1}}) - u(\underbrace{-t-3}_{\substack{-t>3 \\ t<-3}})\right)$$





$$x(\tau) = e^{-\tau+1}\left[u(\tau-1) - u(\tau-3)\right]$$

$$h(t-\tau) = e^{t-\tau+1}\left[u(\underbrace{-(t-\tau)-1}_{\tau > t+1}) - u(\underbrace{-(t-\tau)-3}_{\tau > 3+t})\right)$$



$t+3 < 1: \quad y(t) = 0$

$1 \le t+3 < 3: \quad y(t) = \int_1^{t+3} e^{-\tau+1}\, e^{t-\tau+1}\, d\tau = e^{t+2}\int_1^{t+3} e^{-2\tau}\, d\tau$

$1 \le t+1 < 3: \quad y(t) = \int_{t+1}^{3} e^{-\tau+1}\, e^{t-\tau+1}\, d\tau = e^{t+1}\int_{t+1}^{3} e^{-2\tau}\, d\tau$

$t+1 > 3: \quad y(t) = 0$

feb 06 2

$$\frac{1}{jt-\alpha} \xleftrightarrow{\mathcal{F}} -2\pi\, e^{-\alpha w} \cdot u(w)$$

a) $$x(t) = \frac{\cos(t)}{(jt-3)(jt-1)} \quad ; \quad TF(x(t))\ ??$$

$$x(t) = \cos(t) \cdot x_1(t) = \frac{1}{2} x_1(t) \cdot e^{jt} + \frac{1}{2} x_1(t)\, e^{-jt}$$

$$X(jw) = \frac{1}{2} X_1(j(w-1)) + \frac{1}{2} X_1(j(w+1))$$

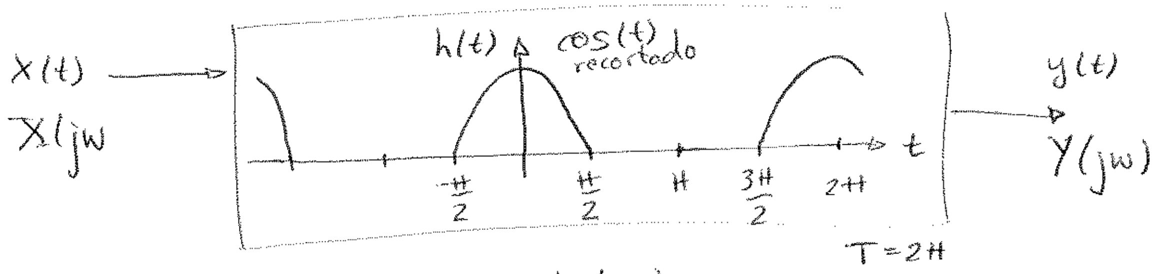
$$x_1(t) = \frac{1}{(jt-3)(jt-1)} \longleftrightarrow X_1(jw)\ ??$$

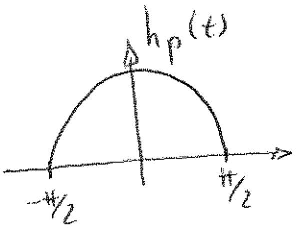
$$x_1(t) = \frac{A}{jt-3} + \frac{B}{jt-1} \longleftrightarrow X_1(jw) = -A\, 2\pi e^{-3w} u(w) + -B\, 2\pi e^{-w} u(w)$$

$$A = x_1(t) \cdot (jt-3) = \left.\frac{1}{jt-1}\right|_{jt=3}$$

$$B = x_1(t)\,(jt-1) = \left.\frac{1}{jt-3}\right|_{jt=1}$$

b)





$$a_K = \left.\frac{H_p(jw)}{2\pi}\right|_{w=k}$$

$$H_p(jw) = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} e^{jt} + \frac{1}{2} e^{-jt}\right) e^{-jwt}\, dt$$

$$Y(jw) = X(jw) \cdot Y(jw) = X(jw) \cdot \sum 2\pi a_K\, \delta\left(w - k\frac{2\pi}{T}\right) =$$

$$= 2\pi \sum \underbrace{a_K\, X(jk)}_{b_k}\, \delta(w-k)$$

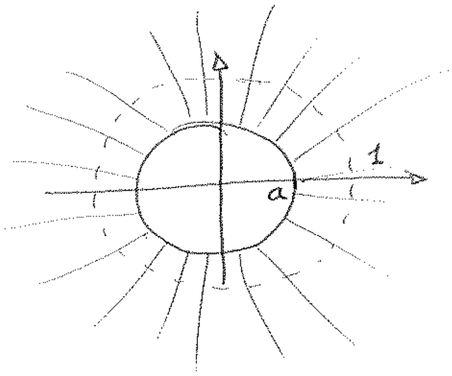
feb 06 3

$$x[n] = a^n \cos(\omega_0 n)\, u[n] \longleftrightarrow X(z) = \frac{1 - a\cos(\omega_0) z^{-1}}{1 - 2a\cos(\omega_0) z^{-1} + a^2 z^{-2}}$$

$$x[n] = \frac{1}{2} \underbrace{a^n e^{j\omega_0 n}}_{(a e^{j\omega_0})^n} u[n] + \frac{1}{2} \underbrace{a^n e^{-j\omega_0 n}}_{(a e^{-j\omega_0})^n} u[n]$$

sabiendo que: $b^n u[n] \longleftrightarrow \frac{1}{1 - b z^{-1}}$, $|z| > b$

$$X(z) = \frac{1/2}{1 - a e^{j\omega_0} \cdot z^{-1}} + \frac{1/2}{1 - a e^{-j\omega_0} z^{-1}}$$



$$X(e^{j\omega}) = X(z)\Big|_{z = e^{j\omega}}$$

b)

$X(z) \longrightarrow$ $\boxed{H(z)?\ h(t)?}$ $\longrightarrow y[n] = \delta[n - n_0]$

$$Y(z) = z^{-n_0} = H(z) \cdot X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = z^{-n_0} \cdot \frac{1 - 2a\cos(\omega_0) z^{-1} + a^2 z^{-2}}{1 - a\cos(\omega_0) z^{-1}}$$

podríamos dividir polinomios y desarrollar pero...

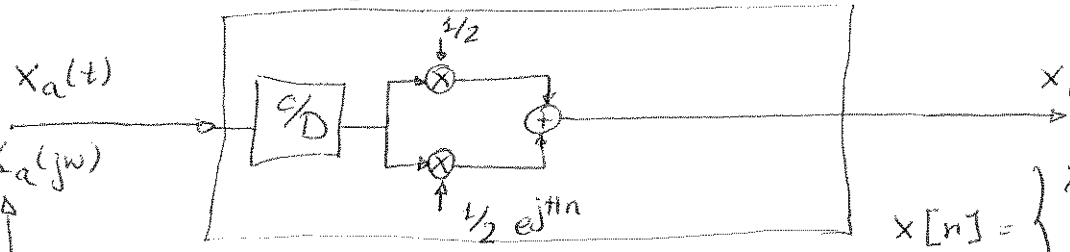
$$H(z) = H_1(z) \cdot z^{-n_0} - 2a\cos(\omega_0) z^{-1} \cdot H_1(z) \cdot z^{-(n_0+1)} + a^2 H_1(z) \cdot z^{-(n_0+2)}$$

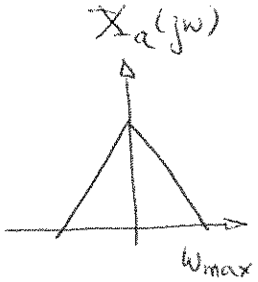
$$h[n] = h_1[n - n_0] - 2a\cos(\omega_0) \cdot h_1[n - (n_0+1)] + a^2 h_1[n - (n_0+2)]$$

$$h_1[n] = (a \cdot \cos(\omega_0))^n \cdot u[n]$$

feb 06 — 4

C/D con anulación de muestras



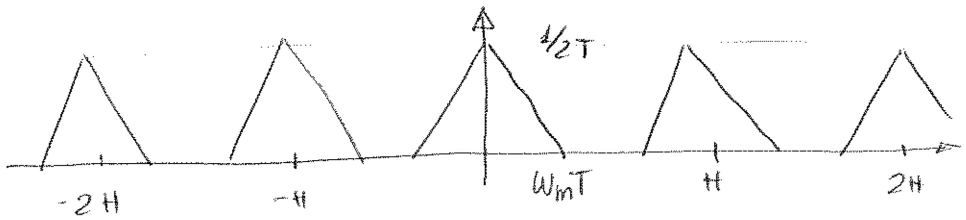


$$x[n] = \begin{cases} x_a(nT), & n \text{ par} \\ 0, & n \text{ impar} \end{cases}$$

$$x[n] = \frac{1}{2}\left(1 + (-1)^n\right) \cdot x_a(nT)$$

$$X_1(e^{j\Omega}) = \frac{1}{T} \sum X_a\left(j\left(\frac{\Omega - k2\pi}{T}\right)\right)$$

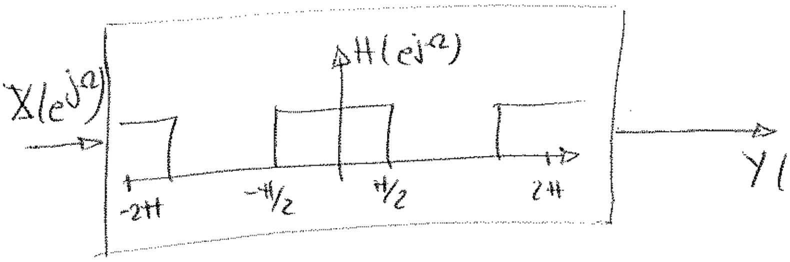
$$X(e^{j\Omega}) = \frac{1}{2T} \sum X_a\left(j\left(\frac{\Omega - k2\pi}{T}\right)\right) + X_a\left(j\left(\frac{\Omega - \pi - k2\pi}{T}\right)\right) = \frac{1}{2T}\sum X_a\left(j\frac{\Omega - k\pi}{T}\right)$$



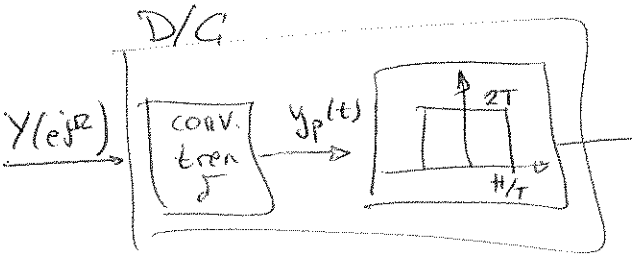
$$\pi - w_m T > w_m T$$

$$\frac{2\pi}{T} > 4 w_m$$

$$w_s > 4 w_m$$



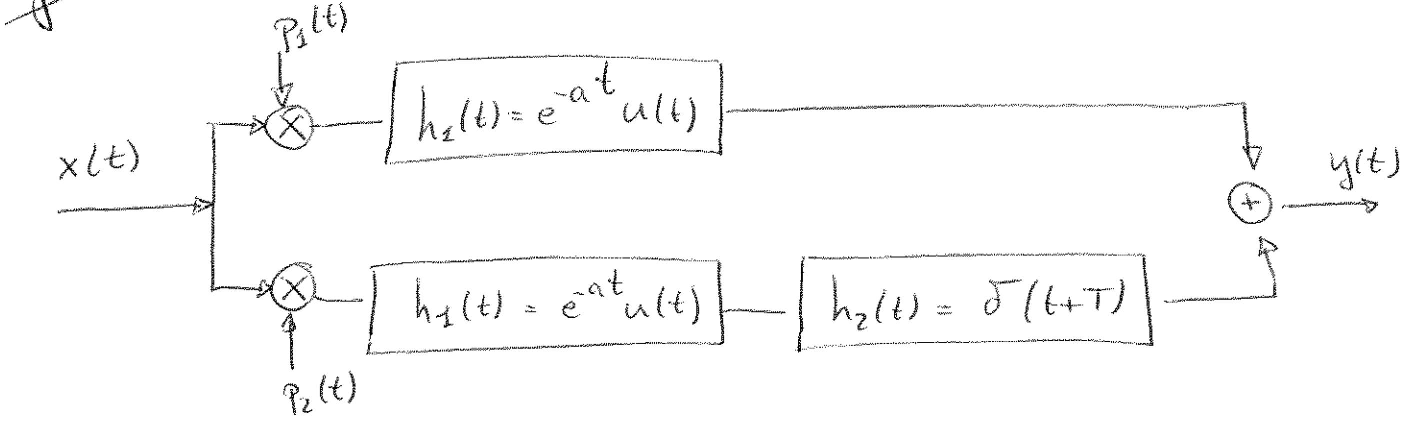
$$Y(e^{j\Omega}) = \frac{1}{2T} \sum X_a\left(j\,\frac{\Omega - k2\pi}{T}\right)$$

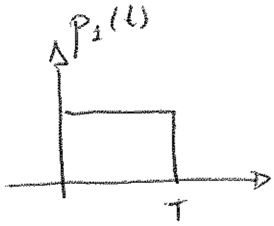


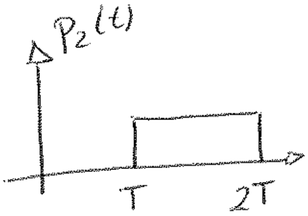
$$y(t) = y_p(t) * h_i(t) = \sum y[n]\, h_i(t - nT)$$

$$y_p(t) = \sum y[n]\, \delta(t - nT)$$

jun 2004 1







a) $y(t)\Big|_{x(t)=\delta(t-t_0)}$??

$t_0 \in (0,T) \rightarrow y(t) = \delta(t-t_0) * h_1(t) = h_1(t-t_0)$

$t_0 \in (T,2T) \rightarrow y(t) = \delta(t-t_0) * h_1(t) * h_2(t) = h_1(t+T-t_0)$

lineal: $\Big(\big(\alpha\, x_1(t) + \beta\, x_2(t)\big) \cdot p_1(t)\Big) * h_1(t) +$

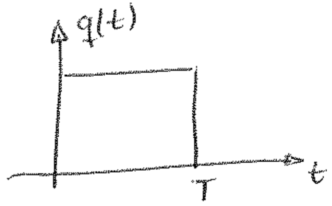
$+ \Big(\big(\alpha\, x_1(t) + \beta\, x_2(t)\big) \cdot p_2(t)\Big) * h_1(t) * h_2(t)$ <u>si lineal</u>

invariante: No invariante ⇒ variante
(desplazamientos en la entrada pasan por diferentes sistemas ⇒ diferentes salidas)

causalidad: No causal ($h_2(t)$)

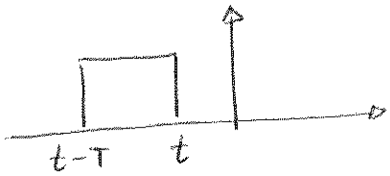
estabilidad: sí estable

b)



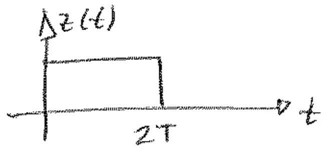
$y_q(t)\Big|_{x(t)=q(t)}$??

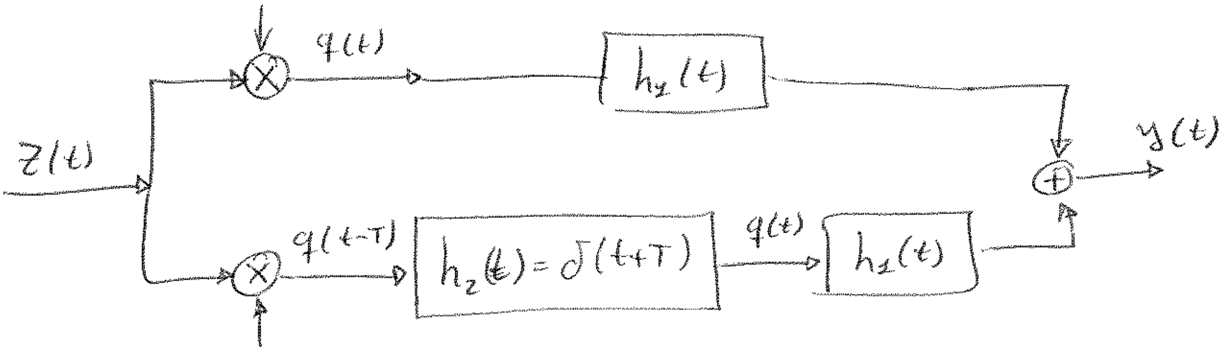
$$y_q(t) = q(t) * h_1(t) = \int h_1(\tau)\, q(t-\tau)\, d\tau$$



$$y_q(t) = \begin{cases} t<0 & y_q(t) = 0 \\ 0<t<T & y_q(t) = \int_0^T e^{-a\tau}\, d\tau \\ t>T & y_q(t) = \int_{t-T}^{t} e^{-a\tau}\, d\tau \end{cases}$$

c) $y_z(t)\Big|_{x(t)=z(t)}$??





$$y_z(t) = 2\left(q(t) * h_1(t)\right) = 2 \cdot y_q(t)$$

d) $y_u(t)\Big|_{x(t)=u(t)}$??

$$y_u(t) = y_z(t) = 2 \cdot y_q(t)$$

Jun 2004 2

LTI causal

$$x(t) \longrightarrow \boxed{\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} + x(t)} \longrightarrow y(t)$$

$$Y(s)\cdot s^2 + Y(s)\cdot s + Y(s) = X(s)\cdot s^2 - X(s)\cdot s + X(s)$$

$$H(s) = \frac{s^2 - s + 1}{s^2 + s + 1} \Rightarrow s_{polo} \; -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} = \sigma_1 \pm j w_1$$

$$\text{causal} \Rightarrow H(jw) = H(s = jw) = \frac{-w^2 - jw + 1}{-w^2 + jw + 1}$$

$$|H(jw)| = \frac{|-w^2 + 1 - jw|}{|-w^2 + 1 + jw|} = 1$$

b)

$$\begin{array}{l|l} s^2 - s + 1 & s^2 + s + 1 \\ \hline -s^2 - s - 1 & 1 \\ \hline 0 \; -2s + 0 & \end{array}$$

$$H(s) = 1 - 2\frac{s}{s^2 + s + 1}$$

$$h(t) = \delta(t) - 2\mathcal{L}^{-1}\Big(\underbrace{\frac{s}{s^2 + s + 1}}_{H_1(s)}\Big)$$

$$H_1(s) = \frac{s}{s^2 + s + 1} = \frac{s}{(s - s_1)(s - s_1^*)} = \frac{A}{s - s_1} + \frac{B}{s - s_1^*}$$

$$A = H_1(s)\cdot(s - s_1)\Big|_{s = s_1} =$$

$$B = H_1(s)\,(s - s_1^*)\Big|_{s = s_1^*} = A^*$$

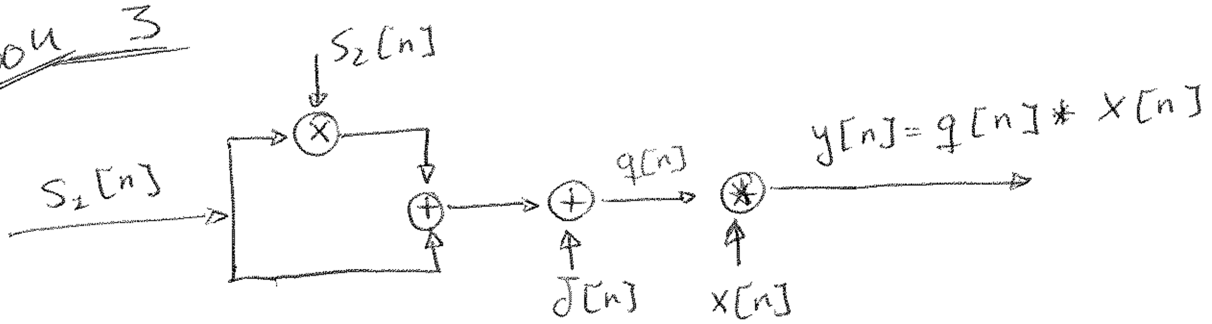
$$h_1(t) = A\cdot e^{s_1 t}u(t) + A^* e^{s_1^* t} u(t) = |A| e^{j\theta} e^{\sigma_1 t} e^{jw_1 t} u(t)$$

$$+ |A| e^{-j\theta} e^{\sigma_1 t} e^{-jw_1 t} u(t) = 2|A| e^{\sigma_1 t} \cos(w_1 t + \theta)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} \underbrace{a_k}_{\text{datos}} e^{jk\omega_0 t} \quad ; \quad y(t) = \sum \underbrace{b_k}_{\hookrightarrow\, a_k \cdot H(jk\omega_0)} e^{jk\omega_0 t}$$

$$P_y = \frac{1}{T}\int_T y(t)\, dt = \sum_{k=-\infty}^{+\infty} |b_k|^2 = \sum_{k=-\infty}^{+\infty} |a_k|^2 \, |H(jk\omega_0)|^2 \neq P_x$$

jun2004 3



$$s_1[n] = \frac{\operatorname{sen}\left(\frac{\pi}{6} n\right)}{\pi n}$$

$$s_2[n] = 2 \cos\left(\frac{\pi}{2} n\right)$$

a) $q[n] = \delta[n] - s_1[n] \cdot s_2[n] - s_1[n]$

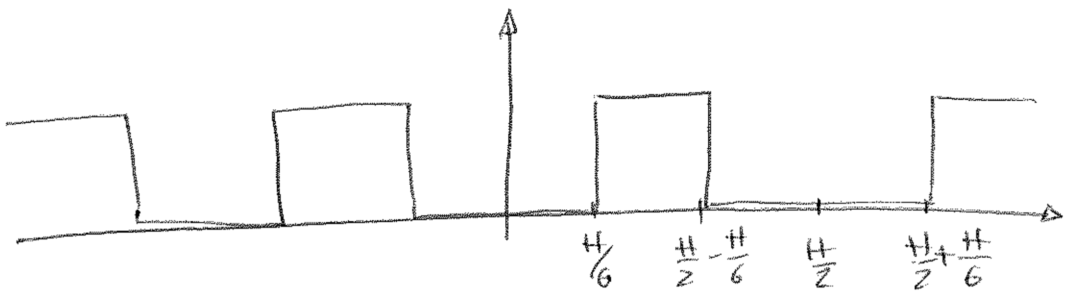
$$Q(e^{j\omega}) = 1 - (\text{---------})$$

$$S_1(e^{j\omega}) = \begin{cases} 1 & |\omega| < \pi/6 \\ 0 & \pi/6 < |\omega| < \pi \end{cases} \quad \text{periódica } 2\pi$$

$$s_1[n] \cdot s_2[n] = s_1[n] \cdot \left(e^{j\frac{\pi}{2} n} + e^{-j\pi/2\, n} \right)$$

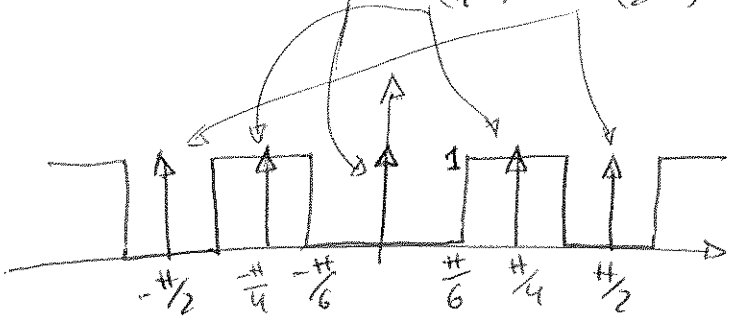
$$TF(s_1[n] \cdot s_2[n]) = S_1\left(e^{j(\omega - \pi/2)}\right) + S_1\left(e^{j(\omega + \pi/2)}\right)$$

$$Q(e^{j\omega}) = 1 - S_1\left(e^{j(\omega - \pi/2)}\right) - S_1\left(e^{j(\omega + \pi/2)}\right) - S_1(e^{j\omega})$$



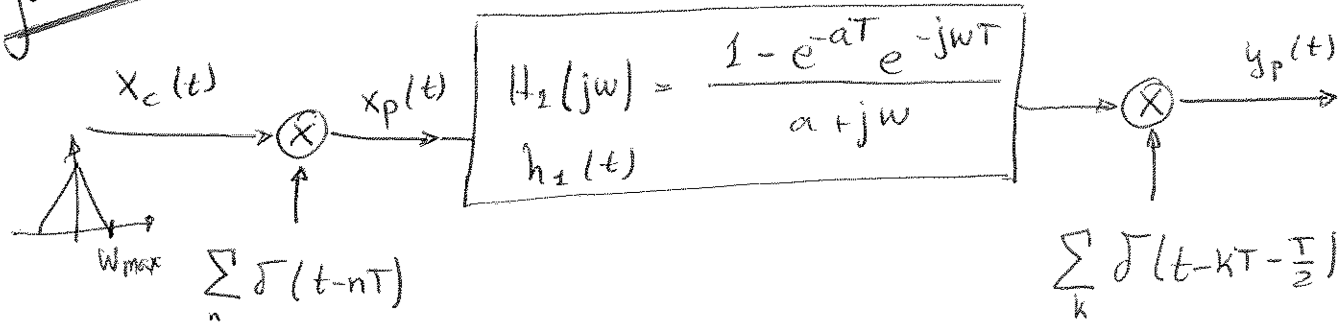
b) $y[n]\Big|_{x[n] = 1+\cos\left(\frac{\pi}{4}n\right)+\text{sen}\left(\frac{\pi}{2}n\right)} = y[n]\Big|_{x[n]=\cos\left(\frac{\pi}{4}n\right)} =$

$= \cos\left(\frac{\pi}{4}n\right)$



c) $y[n]\Big|_{x[n] = \frac{\text{sen}\left(\frac{\pi}{24}n\right)}{\pi n}\cdot\cos\left(\frac{\pi}{4}n\right)} = x[n]$

jun 2004 4



$$H_1(jw) = \frac{1}{a+jw} - e^{-aT}\frac{1}{a+jw}e^{-jwT}$$

$$h_1(t) = e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T) = e^{-at}(u(t)-u(t-T))$$

$$x_p(t) = \sum x_c(nT)\cdot\delta(t-nT)$$

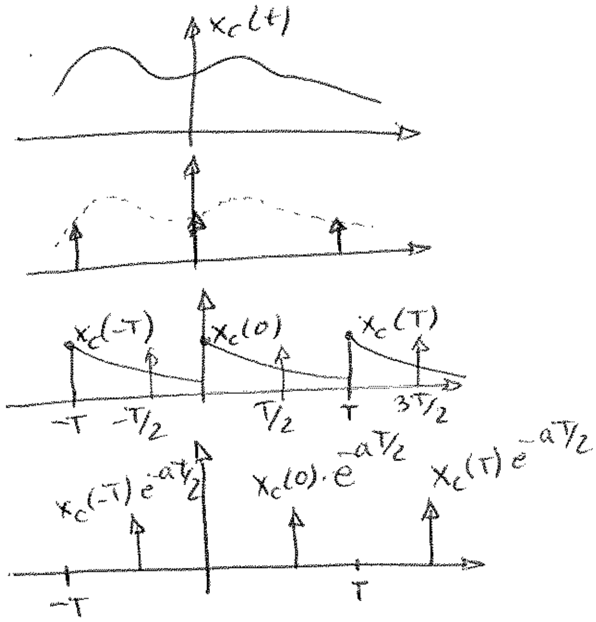
$$y_c(t) = \sum x_c(nT)\cdot h_1(t-nT)$$

$$y_p(t) = y_c(t)\cdot\sum\delta(t-kT-T/2) =$$

$$= \sum x_c(nT)h_1(t-nT)\delta(t-nT-T/2)$$

$$= h_1(T/2)\sum x_c(nT)\delta(t-nT-T/2)$$

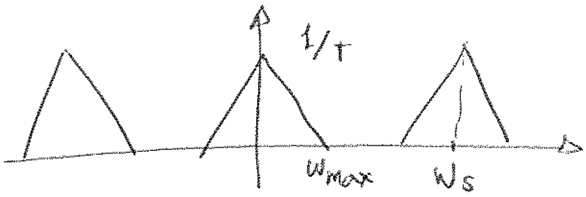
$$= h_1(T/2)\cdot x_p(t-T/2)$$



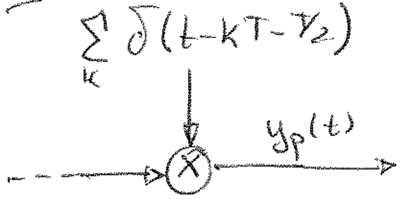
b)

$$Y_p(j\omega) = e^{-aT/2}\, e^{-j\omega T/2}\, X_p(j\omega) =$$

$$= e^{-(a+j\omega)T/2} \cdot \sum \frac{1}{T} X_c(j(\omega - k\omega_s))$$



c)



$$H_i(j\omega)\cdot e^{-(a+j\omega)T/2} = \begin{cases} T, & |\omega| < \omega_s/2 \\ 0, & \text{resto} \end{cases}$$

$$H_i(j\omega) = \begin{cases} T\, e^{(a+j\omega)T/2}, & |\omega| < \omega_s/2 \\ 0, & \text{resto} \end{cases}$$

d)